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Stability estimate for the Helmholtz equation with rapidly jumping coefficients *

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August 31, 2018

Abstract

The goal of this paper is to investigate the stability of the Helmholtz equation in the high-frequency regime with non-smooth and rapidly oscillating coefficients on bounded domains. Existence and uniqueness of the problem can be proved using the unique continuation principle in Fredholm's alternative. However, this approach does not give directly a coefficient-explicit energy estimate. We present a new theoretical approach for the one-dimensional problem and find that for a new class of coefficients, including coefficients with an arbitrary number of discontinuities, the stability constant (i.e., the norm of the solution operator) is bounded by a term independent of the number of jumps. We emphasize that no periodicity of the coefficients is required. By selecting the wave speed function in a certain "resonant" way, we construct a class of oscillatory configurations, such that the stability constant grows exponentially in the frequency. This shows that our estimates are sharp.

Keywords. Helmholtz equation, high frequency, heterogeneous media, stability estimates

MSC 2010. Primary: 65N80, 65N12, 35B35; Secondary: 35J05.

1 Introduction

High-frequency scattering problems have many important applications which include, e.g., radar and sonar detection as well as medical and seismic imaging. In physics, such problems are studied intensively in the context of wave scattering in disordered media and localization of waves with the goal to *design* waves with prescribed intensity, interference, localized foci, parity-time symmetry, etc. Important applications are in nano photonics and lasers – see, e.g., [2], [15], [31], [16], [19], [27], [17] for references to the theoretical and experimental physics literature.

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Their efficient and reliable numerical modelling is a challenge and the development of fast numerical methods is far from being mature. Such problems are often modelled in the frequency domain, where a time-periodic ansatz is employed for the wave equation which, in turn, results in the Helmholtz equation in the high-frequency regime, i.e., with large wave number. Moreover, in applications such as seismic or medical imaging, the media typically are heterogeneous and, consequently, the coefficients in the Helmholtz equation become variable. The numerical analysis for these types of problems is much less developed as the high-frequency homogeneous case.

In this paper, we discuss the Helmholtz equation of the form

$$- \operatorname{div}(a \text{ grad } u) - \left(\frac{\omega}{c}\right)^2 u = f \text{ in } \Omega, \tag{1a}$$

on a bounded Lipschitz domain Ω, frequency $\omega > 0$, positive wave speed c and diffusion coefficient a, where both, c and a, are variable. On the Helmholtz problem, we impose impedance boundary conditions

$$a \frac{\partial u}{\partial \mathbf{n}} - \mathrm{i} \sqrt{a} \frac{\omega}{c} u = g \text{ on } \Gamma = \partial\Omega. \tag{1b}$$

Let $L^2(\Omega)$ be the usual Lebesgue space with scalar product and norm

$$(u, v) := \int_\Omega u\bar{v}, \qquad \|u\| := (u, u)^2.$$

We define the "energy space" by $\mathcal{H} := H^1(\Omega)$, equipped with

$$\|w\|_{\mathcal{H}} := \sqrt{\left\|\sqrt{a}\nabla w\right\|^2 + \left\|\frac{\omega}{c} w\right\|^2}. \tag{2}$$

The variational formulation of (1) is to find $u \in \mathcal{H}$ such that

$$B(u, v) := (a\nabla u, \nabla v) - \left(\frac{\omega}{c} u, \frac{\omega}{c} v\right) - \mathrm{i}\left(\sqrt{a}\frac{\omega}{c} u, v\right)_{L^2(\Gamma)} = (f, v) + (g, v)_{L^2(\Gamma)} =: F(v) \quad \forall v \in \mathcal{H}. \tag{3}$$

To solve the problem numerically one may use abstract, conforming Galerkin methods, i.e., choose a finite-dimensional subspace $S \subset \mathcal{H}$ and seek for $u_S \in S$ such that

$$B(u_S, v) = F(v) \qquad \forall v \in S. \tag{4}$$

The stability constant C_{stab} of the problem, satisfying

$$\|u\|_{\mathcal{H}} \leq C_{\text{stab}} \left(\|f\|^2 + \|g\|^2_{H^{1/2}(\Gamma)}\right)^{\frac{1}{2}},$$

plays an important role for the design and the numerical analysis of abstract Galerkin methods of the form (4) and much research has been devoted to estimate this constant. C_{stab} in general depends on the wave number and the coefficient functions a and c.

In two dimensions and for a, c positive constants, Ω convex or C^1-star shaped the stability constant $C_{\text{stab}} > 0$ is independent of f, g, and ω [20]. The result was extended to higher dimensions in [9]. For the case of general Lipschitz domains and constant coefficients, it was proved in [10, Theorem 2.4] that $C_{\text{stab}} \leq C\omega^{5/2}$ while the result was improved in [32] to $C_{\text{stab}} \leq C\omega$. For bounded domains with C^∞ boundaries it was proved in [4, Theorem 1.8] that $C_{\text{stab}} = O(1)$. The convergence analysis for hp-Galerkin finite elements as in [21, 22] shows that in the error estimate C_{stab} is multiplied by a term, which is exponentially small in the polynomial degree p, so that a choice $p \sim \log(\omega)$ preserves the optimal convergence order without pollution.

Stability Estimates for the heterogeneous Helmholtz Problem. The theory of the heterogeneous Helmholtz equation is much less developed than the one for constant coefficients since the wave strongly depends on interfaces, variations of the coefficients, and can exhibit localization, interferences, complicated variations of intensity, in particular, if scattering through disordered media is considered, see, e.g., [31], [23].

First results regarding rigorous stability estimates in one dimension go back to [3], where c is supposed to be sufficiently smooth (at least C^1). For $d \geq 2$ and slowly varying coefficients, similar results were proved in [26] and [12]. The main theoretical approaches for estimating the stability in the case of varying coefficients are as follows.

a) "Rellich" or "Morawetz" multipliers. For general dimension a test function of the form $v = x.\nabla u$ and $v = u$, or modifications thereof is employed in the variational formulation of the Helmholtz equation which allows to estimate the L^2-part of the solution in terms of its H^1-semi norm and the data. In turn, this estimate is used to estimate the stability constant. This technique was further developped for the heterogeneous Helmholtz equation in [26].

In [8, 12], stability is achieved via the "Rellich trick", by replacing the smooth field x with a coefficient-dependent piecewise smooth field. It turns out that, only oscillatory wave speeds which are small perturbations of a constant wave speed, can be handled by this technique .

b) Full space methods. If heterogeneous Helmholtz problems are considered in full space, methods from semi-classical/asymptotic/microlocal analysis can be applied and estimates for the stability constant are derived, e.g., in [7] for smoothly varying coefficients and in [5] for a full space problem with one inclusion and a discontinuity across the interface. It is shown that for these cases, the stability constant can grow *at most* exponentially in ω. In [29], and [28], [23], examples are presented for smooth coefficients/discontinuous coefficients with one interface where the stability constant grows super-algebraically.

c) Homogenization. For *periodic*, heterogeneous media, methods of homogenization can be applied (for diameters of inclusions tend to zero and their number goes to infinity while the frequency ω is assumed to be fixed) to derive effective equations (see, e.g., [6], [25]) which then can be analysed by methods for Helmholtz equations with constant coefficients.

d) Matrix techniques. In layered materials, one can approximate the wave propagation problem by restricting to a linear combination of *finitely* many waves types (longitudinal and shear waves) in each homogeneous material part of the domain as an ansatz and combine this across the interfaces by transmission and boundary conditions (see, e.g., [17] and citations therein). This leads to a linear algebraic system for the coefficients in this ansatz. Our approach is following the same basic idea. However, since we are in 1D the fundamental system of the Helmholtz equation consists of only two types of waves (one is outgoing, the other one ingoing) and we end up with an $n \times n$ tridiagonal block system consisting of 2×2 matrices per block – here n denotes the number of discontinuities. Our main achievement in this paper is the derivation of a stability bound of the corresponding Green's function which is explicit with respect to the wave number, the coefficients, and the number of jumps. In the case of a stochastic 1D medium the Green's function has been analysed in [11] and asymptotic properties have been derived in a stochastic setting. Also in this case, realizations of the random coefficients are considered where the wave exhibits localization and near-resonances.

Explicit Estimates in 1D for the Heterogeneous Helmholtz Problem. In contrast to the Green's function (of the ordinary differential equation) for the Helmholtz problem in one dimension with constant coefficients, the Green's function in the case of variable coefficients is not known explicitly and more subtle mathematical tools have to be employed for 1D stability estimates. In [3], a one-dimensional model problem for the Helmholtz problem was considered and well-posedness of (1) was proved for positive and bounded wave speed c. A test function of Rellich/Morawetz-type (cf. [30], [24]) of the form $v = au' + bu$ for some functions a and b (chosen as solutions of a certain ODE) has been employed to prove the wave-number independent stability bounds for $c \in C^1(\Omega)$ while the constant depends on $\|c'\|_\infty$. In [18], [13], the test function $v(x) = xu'(x)$ of Rellich/Morawetz-type has been used to prove stability bounds for a certain fluid-solid interaction problem for elastic waves which can be considered as a problem with a wave speed which has discontinuities at two points. For the one-dimensional problem, some explicit estimates of the stability constant were published only recently. In the thesis [8], a stability estimate for the case of, possibly, very large numbers of layers and general, layer-wise constant wave speed c and $a = 1$ is proved. The estimates are explicit in the number of jumps and the values of the coefficients. In the worst case, the estimate grows exponentially in the number of interfaces. A generalized result is proved in [12], where a, c are both piecewise monotonic functions with bounded variation. In this paper, we present estimates of the stability constant in 1D for certain classes of piecewise constant wave speeds $c \in L^\infty(\Omega, \mathbb{R}_{>0})$ which are explicit in the wave number ω and c. We do not require any periodicity on c. This result is finer than the result in [12], where the stability constant grows exponentially in the variation of c and a. In addition, we will construct a class of configurations which show that a) our estimates are sharp and b) that these configurations are very rare.

Outline of the Paper. Since the proofs in this paper are quite technical we explain our approach and the main results here as an outline and summary. In Section 2, we discuss briefly the well-posedness of the problem in the multidimensional case and state a conjecture on the behaviour of the stability constant C_{stab} for a wave speed $c \in L^\infty(\Omega, \mathbb{R}_{>0})$. We establish results towards this conjecture in 1D in the following sections. In Section 3, we consider the Helmholtz equation (with piecewise constant wave speed c, inhomogeneous impedance boundary conditions, and without volume forces) and employ an ansatz as a linear combination of ingoing and outgoing waves on each subinterval combined by transmission and boundary conditions. This leads to a linear $n \times n$ tridiagonal block system for the coefficients in this ansatz. The blocks are symmetric (but not Hermitian) 2×2 matrices and n denotes the number of jumps of the wave speed. We derive a simple representation (see Corollary 8 for details) of the corresponding Green's function:

$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n-2m+1,2n}\right| = \left|\prod_{\ell=n-m+1}^{n} \frac{\sqrt{1-q_\ell^2}}{(1+q_\ell Q_{\ell-1})}\right|, \qquad 1 \le m \le n,$$

where $q_\ell = \frac{c_{\ell+1}-c_\ell}{c_{\ell+1}+c_\ell}$ denote the relative jumps on neighbouring intervals and (Q_ℓ) is a recursively defined sequence with values in the interior of the complex unit disc. This representation is based on Cramer's rule and a recursive formula for the arising determinants. The proofs of these representations are very technical but elementary and we have shifted them to the last section (§6). This representation allows us to reduce the estimate of the solution operator to an analysis of the sequence Q_ℓ, depending on the piecewise constant values of the wave speed c and a phase factor σ_ℓ which encodes the interplay of the wave number ω, the values of the wave speed c, and the width of the subinterval τ_ℓ. An estimate of the entries of Green's function is given in Lemma 9

$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n-2m+1,2n}\right| \le \frac{1}{\sqrt{1-|Q_{n-m}|^2}}, \qquad 1 \le m \le n$$

and we recall $|Q_\ell| < 1$. From this, we derive our main stability estimate (Theorem 10)

$$\left(\int_\Omega |u'|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2\right)^{\frac{1}{2}} \leq C_{\text{stab}} \max\{|g_1|, |g_2|\} \quad \text{with} \quad C_{\text{stab}} := 4\frac{c_{\max}}{c_{\min}} \max_{1\leq\ell\leq n} \frac{1}{\sqrt{1-|Q_\ell|^2}}. \quad (5)$$

This shows the importance of the sequence $(Q_\ell)_\ell$. However, this sequence depends on a very high-dimensional parameter space involving the location of the n jump points, the wave number ω, and the values of the wave speed. Our focus is on large number of jumps n and hence, the analysis of Q_ℓ becomes quite involved.

Before proving general estimates on Q_ℓ we present two classes of examples in Section 4. Both examples have in common that the number of jumps tends to infinity and known *a priori bounds* for the stability constant (see, e.g., [8], [12]) tend to infinity for these cases. However, the near-resonance cases seem to be very rare and their construction require a subtle tuning of the parameters (§4.2) so that $|Q_\ell|$ approaches, exponentially fast, the value 1 and, in turn, C_{stab} grows exponentially with respect to ω (cf. (5), Remark 14). The construction of *well-behaved* examples is much simpler (§4.1); although the bounds in [8], [12] tend to infinity as $n \to \infty$, our new bound (and of course the solution) stays bounded independent of n. The implications are two-fold: the results in [8], [12] are sharp in general, but very pessimistic in many parameter configurations.

Section 5 is devoted to the parameter-explicit estimate of the sequence $(Q_\ell)_\ell$. In Lemma 15 we employ tools from complex analysis to maximize the sequence with respect to all parameters and derive

$$|Q_{\ell-1}| \leq \frac{1-\kappa^{-\ell}}{1+\kappa^{-\ell}} \quad \text{with the \textit{condition number} of the wave speed } \kappa := \|c\|_{\max} \left\|c^{-1}\right\|_{\max}. \quad (6)$$

Hence, if the number of jumps are bounded from above by $O(\omega)$, the stability constant in (5) grows at most exponentially in ω (see Section 5.2). The analysis of the remaining case, i.e., when the number of jumps is much larger than ω, is more involved and we restrict to a slightly simplified parameter setting in Section 5.3. We still allow the jump points to be distributed in an arbitrary way while we assume that the wave speed is jumping between only two positive values. Then it is again possible (Prop. 19) to maximize $|Q_\ell|$ for this setting. By employing the side condition that the lengths of the subinterval with constant wave speed have to sum up to the total length of Ω we derive an estimate of $|Q_\ell|$ of the form

$$|Q_\ell| \leq 1 - C\alpha^{\omega/c_{\min}}$$

for some $\alpha \in (0, 1)$. In combination with (5) the conjectured bound in the considered scenarios follows (§5.4).

2 High-Frequency Helmholtz Equations with Variable Coefficients

2.1 Helmholtz Equation for Varying Coefficients

We consider the Helmholtz Equation on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with variable wave speed c and diffusion coefficient a

$$-\operatorname{div}(a \text{ grad } u) - \left(\frac{\omega}{c}\right)^2 u = f \text{ in } \Omega. \quad (7)$$

The right-hand side f is in $L^2(\Omega)$ and we denote by ω the frequency parameter, bounded from below by $\omega_0 > 0$. We assume that the wave speed as well as the diffusion coefficient are bounded in the following way

$$\begin{aligned} 0 < c_{\min} \leq & c \leq c_{\max} < \infty, \\ 0 < a_{\min} \leq & a \leq a_{\max} < \infty. \end{aligned}$$

We denote the boundary of Ω by $\Gamma := \partial\Omega$ and let Γ_{R}, Γ_{D} be relatively open pairwise disjoint subsets of Γ, with $\Gamma = \overline{\Gamma_{\mathrm{R}} \cup \Gamma_{\mathrm{D}}}$. On (7), we impose the impedance boundary conditions

$$a \frac{\partial u}{\partial \mathbf{n}} - \mathrm{i}\sqrt{a}\beta u = g \text{ on } \Gamma_{\mathrm{R}}, \quad u = 0 \text{ on } \Gamma_{\mathrm{D}}, \tag{8}$$

for a given impedance coefficient $\beta \in L^\infty(\Gamma_{\mathrm{R}}, [0, \beta_{\max}])$, $\beta_{\max} < \infty$. Let $\mathcal{H} := \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}$, with norm defined in (2). The weak formulation of the problem is: For given $\mathcal{F} \in \mathcal{H}'$, find $u \in \mathcal{H}$ such that

$$B(u, v) = \mathcal{F}(v) \quad \forall v \in \mathcal{H}, \tag{9}$$

where $B(\cdot, \cdot)$ is defined as in (3).

2.2 Well-Posedness

Theorem 1 *Let $\beta \in L^\infty(\Gamma_{\mathrm{R}}, \mathbb{R}_{\geq 0})$ be such that* $\operatorname{supp}\beta$ *has positive $(d-1)$-dimensional boundary measure. Let $a, c \in L^\infty(\Omega)$ with $0 < a_{\min} \leq a^* \leq a_{\max} < \infty$. For $d \geq 3$ we assume, in addition, that $a \in C^{0,1}(\Omega)$. Then, for any $\omega \geq \omega_0$, the heterogeneous Helmholtz problem (9) is well posed.*

For $d = 2$ the proof can be found in [1, Theorem 1.1], for $d = 1$ the proof is very similar. For $d = 3$, one uses the fact that $a \in C^{0,1}(\Omega)$ can be extended to $\mathbb{R}^d$. The proof of the theorem is based on Fredholm's alternative (cf. [1, 12, 14]). This technique, however, does only provide an implicit stability estimate. It is therefore not straightforward, how the stability constant depends explicitly on ω, c or other parameters.

2.3 Maximal Growth of the Stability Constant

As discussed, there has been several contributions for finding estimates of the stability constant that are explicit in the parameters ω, a and c. Recent results (cf. [8, 12]) show that the stability constant can be bounded with respect to the number of jumps or the total variation of the coefficients, if they are piecewise constant. Considering coefficients that are highly oscillatory and have an increasing number of jumps, the stability constant known from [8, 12] are diverging to infinity. Based on known theoretical results and also numerical experiments, we formulate the following conjecture.

Conjecture 2 *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $a = 1$, $c \in L^\infty(\Omega)$, with $0 < c_{\min} \leq c \leq c_{\max} < \infty$, $\omega \geq \omega_0 > 0$ it holds*

$$\left(\int_\Omega |\nabla u|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2 \right)^{\frac{1}{2}} \leq C_{\mathrm{stab}} \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)} \right)^{\frac{1}{2}}, \tag{10}$$

with

$$C_{\mathrm{stab}} \leq C_1 \exp\left(C_2 \omega\right),$$

$C_1, C_2 > 0$ depending on $c_{\min}$, $c_{\max}$ and Ω.

For constant wave speed c, there is a fairly good knowledge about the constant C_{stab}; under moderate assumptions on the domain it can be shown that C_{stab} only grows algebraically with respect to ω and in many cases this constant is bounded independent of ω (see the literature review in §1).

In higher dimension, there exist much less results. As explained in the literature review (see §1), examples are known where C_{stab} grows at *least* super-algebraically with respect to ω while for smoothly varying coefficients and coefficients with a discontinuity across one interface, an upper bound can be proved which grows *at most* exponentially in ω. All these results are in accordance with our conjecture.

In this paper, we present further results towards this conjecture for a one-dimensional example. We will derive a recursive representation of the Green's function which allows to understand well-behaved parameter configurations as well as to detect near-resonance cases. In turn, this allows us to find classes of wave speeds, where the stability constant grows exponentially with respect to ω.

3 Helmholtz Equation in one Dimension

We subdivide the domain $\Omega = (-1, 1)$ into $(n + 1)$ intervals by introducing the mesh points

$$-1 = x_0 < x_1 < \dots x_{n+1} = 1 \tag{11}$$

and define the subintervals $\tau_j = (x_{j-1}, x_j)$, $1 \leq j \leq n + 1$ and widths

$$h_j := x_j - x_{j-1}.$$

In this section, we consider piecewise constant wave speed which is given by

$$c(x) := c_j \text{ for } x \in \tau_j, \tag{12}$$

where c_j are positive constants. For a positive wave number $\omega \in \mathbb{R}_{\geq\omega_0}$ we consider the homogeneous Helmholtz equation in the strong form

$$\begin{array}{ll} -u'' - \left(\frac{\omega}{c}\right)^2 u = 0 & \text{in } \Omega = (-1,1), \\ -u' - \mathrm{i}\,\frac{\omega}{c_1} u = g_1 & \text{at } x = -1, \\ u' - \mathrm{i}\,\frac{\omega}{c_n} u = g_2 & \text{at } x = 1. \end{array} \tag{13}$$

The physical parameters of the problem are a) the jump points $(x_\ell)_{\ell=1}^n$ satisfying (11), b) the (positive) values c_ℓ, $1 \leq \ell \leq n+1$, of the wave speed on the subintervals, c) the positive wavenumber $\omega > 0$, and d) the right-hand side g_1, g_2 in (13). For the *mathematical* analysis of the problem, we introduce derived parameters (which will simplify the notation in the proofs), namely, phase factors $\sigma_j \in \mathcal{C} := \{z \in \mathbb{C} : |z| = 1\}$ and relative jumps q_j

$$\sigma_j := \exp\left(-2\,\mathrm{i}\,\frac{h_{j+1}\omega}{c_{j+1}}\right), \qquad 0 \leq j \leq n, \tag{14a}$$

$$q_j := \frac{c_{j+1} - c_j}{c_{j+1} + c_j}, \qquad 1 \leq j \leq n. \tag{14b}$$

Remark 3 *Note that $q_j \in [-q, q]$ holds for some $0 < q < 1$ and we have the "inverse" relation*

$$c_{j+1} = c_j \frac{1 + q_j}{1 - q_j}. \tag{15}$$

The phase factor σ_j reflects an interplay between the length of a subinterval, the value of c on the interval, and the wave number ω. This phase factor will be key to the analysis of the growth behaviour of the sequence $(Q_\ell)_\ell$ defined later in (23) and mentioned already in (5).

Variational formulation Let $V := H^1(\Omega)$. Find $u \in V$ such that

$$\int_\Omega \left(u'\bar{v}' - \left(\frac{\omega}{c}\right)^2 u\bar{v} \right) - \mathrm{i} \sum_{x\in\{-1,1\}} \frac{\omega}{c} u(x)\bar{v}(x) = g_1\bar{v}(-1) + g_2\bar{v}(1) \qquad \forall v \in V.$$

3.1 Green's Function and Piecewise Constant Wave Numbers

For $1 \leq j \leq n+1$, we set $u_j := u|_{\tau_j}$. The solution of the homogeneous equation on the interval τ_i can be written in the form

$$u_j(x) = A_j \mathrm{e}^{\mathrm{i}\frac{\omega}{c_j}x} + B_j \mathrm{e}^{-\mathrm{i}\frac{\omega}{c_j}x} \qquad \text{on } \tau_j \tag{16}$$

for A_j, B_j, $1 \leq j \leq n+1$. These coefficients are determined by the transmission conditions that u and u' are continuous at the inner mesh points x_j, $1 \leq j \leq n$, and by the boundary conditions at x_0 and x_{n+1}. The boundary conditions lead to

$$A_1 = \mathrm{i}\frac{c_1}{2\omega}\mathrm{e}^{\mathrm{i}\frac{\omega}{c_1}} g_1 \quad \text{and} \quad B_{n+1} = \mathrm{i}\frac{c_{n+1}}{2\omega}\mathrm{e}^{\mathrm{i}\frac{\omega}{c_{n+1}}} g_2. \tag{17}$$

The remaining coefficients

$$\mathbf{x}^{(2n)} := (B_1, A_2, B_2, \ldots, A_n, B_n, A_{n+1})^\intercal \tag{18}$$

are the solution of a system of linear equations. Let

$$\mathbf{r}^{(2n)} = \frac{\mathrm{i}}{2\omega}\left(\mathrm{e}^{\mathrm{i}\frac{\omega}{c_1}} g_1, 0, \ldots, 0, \mathrm{e}^{\mathrm{i}\frac{\omega}{c_n}} g_2\right)^\intercal \in \mathbb{R}^{2n}. \tag{19}$$

We define the diagonal matrix $\mathbf{D}^{(2n)} \in \mathbb{C}^{(2n)\times(2n)}$ by

$$\mathbf{D}^{(2n)} = \operatorname{diag}\left[\alpha_{1,1}\sqrt{c_1}, \frac{\sqrt{c_2}}{\alpha_{2,1}}, \alpha_{2,2}\sqrt{c_2}, \frac{\sqrt{c_3}}{\alpha_{3,2}}, \ldots, \alpha_{n,n}\sqrt{c_n}, \frac{\sqrt{c_{n+1}}}{\alpha_{n+1,n}}\right], \tag{20}$$

with

$$\alpha_{\ell,j} := \exp\left(\mathrm{i}\frac{\omega}{c_\ell}x_j\right), \qquad 1 \leq j \leq n,\ 1 \leq \ell \leq n+1,$$

and the Green's function

$$\mathbf{M}^{(2n)}_{\text{Green}} := \left(\mathbf{M}^{(2n)}\right)^{-1} = \begin{bmatrix} \mathbf{W}^{(1)} & \mathbf{N}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \left(\mathbf{N}^{(1)}\right)^\intercal & \mathbf{W}^{(2)} & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & & & \mathbf{N}^{(n-1)} \\ \mathbf{0} & \cdots & \mathbf{0} & \left(\mathbf{N}^{(n-1)}\right)^\intercal & \mathbf{W}^{(n)} \end{bmatrix}^{-1} \in \mathbb{C}^{(2n)\times(2n)}. \tag{21}$$

The 2×2 submatrices $\mathbf{W}^{(j)}$ and $\mathbf{N}^{(j)}$ are given by

$$\mathbf{W}^{(j)} := \begin{bmatrix} q_j & \sqrt{1-q_j^2} \\ \sqrt{1-q_j^2} & -q_j \end{bmatrix}, \quad \mathbf{N}^{(j)} := \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sqrt{\sigma_j}} & 0 \end{bmatrix} \text{ with } \sqrt{\sigma_j} := \exp\left(-\mathrm{i}\,\frac{h_{j+1}\omega}{c_{j+1}}\right).$$

Remark 4 $\mathbf{M}^{(2n)}$ *is symmetric, but not Hermitian.*

The derivation of the following lemma can be found in Section 6.

Lemma 5 *The remaining coefficients* $\mathbf{x}^{(2n)}$ *are given by*

$$\mathbf{x}^{(2n)} = \mathbf{D}^{(2n)}\mathbf{M}^{(2n)}_{\mathrm{Green}}\mathbf{D}^{(2n)}\mathbf{r}^{(2n)}. \tag{22}$$

Remark 6 *The matrix* $\mathbf{W}^{(i)}$ *corresponds to a reflection; it holds* $\left(\mathbf{W}^{(i)}\right)^2 = \mathbf{I}$ *and the eigenvalues of* $\mathbf{W}^{(i)}$ *are* -1 *and* 1.

For later use, we define the reduced matrix $\mathbf{M}^{(2n-1)}$ which arises by removing the last row and last column of $\mathbf{M}^{(2n)}$. Explicitly, it holds

$$\mathbf{M}^{(2n-1)} := \begin{bmatrix} \mathbf{M}^{(2n-2)} & \mathbf{b}^{(2n-2)} \\ \left(\mathbf{b}^{(2n-2)}\right)^{\intercal} & q_n \end{bmatrix} \quad \text{with} \quad \mathbf{b}^{(2n-2)} := \left(0, 0, \ldots, 0, -\frac{1}{\sqrt{\sigma_{n-1}}}\right)^{\intercal}.$$

One key role for the analysis of the solution operator of (13) is played by the derivation of a representation of the entries of $\mathbf{M}^{(2n)}_{\mathrm{Green}}$. We denote by $\mathbf{M}^{(2n,i,j)}$ the matrix which arises if the i-th row and the j-th column of $\mathbf{M}^{(2n)}$ are removed. According to Cramer's rule we have

$$\left(\mathbf{M}^{(2n)}_{\mathrm{Green}}\right)_{i,j} = (-1)^{i+j}\,\frac{\det \mathbf{M}^{(2n,i,j)}}{\det \mathbf{M}^{(2n)}}.$$

To express the determinant of $\mathbf{M}^{(2n)}$ as a product of terms with positive modulus we introduce some notation. We define the sequence Q_m by the recursion

$$\begin{aligned} Q_1 &= \tfrac{q_1}{\sigma_1}, \\ Q_j &= \frac{q_j + Q_{j-1}}{\sigma_j\left(1 + q_j Q_{j-1}\right)}, \quad 2 \le j \le n. \end{aligned} \tag{23a}$$

For later use, we define the quantity Q'_j of same modulus by

$$Q'_j := \sigma_j Q_j \tag{23b}$$

Formally, we set $Q_0 = 0$. Note that Q'_j is independent of σ_j and Q_j depends on $(\sigma_i)_{i=1}^{j}$, $(q_i)_{i=1}^{j}$. This allows to define

$$\tilde{p}_n(\boldsymbol{\sigma}, \mathbf{q}) := \prod\nolimits_{j=1}^{n-1}\left(1 + q_{j+1}Q_j\right) \tag{24}$$

Here, $\boldsymbol{\sigma} = (\sigma_j)_{j=1}^{n-1}$ and $\mathbf{q} = (q_j)_{j=1}^{n}$ in $\tilde{p}_n$. The proof of the following lemma is postponed to Section 6.

Lemma 7 *For $n \geq 2$, it holds*

$$\det \mathbf{M}^{(2n)} = (-1)^n \tilde{p}_n,$$
$$\det \mathbf{M}^{(2n-1)} = -\sigma_n Q_n \det \mathbf{M}^{(2n)}.$$

In view of the special structure of the right-hand side (19), we are particularly interested in the first and last column of $\mathbf{M}_{\text{Green}}^{(2n)}$, denoted by $\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{*,1}$, $\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{*,2n}$. We investigate the last column, i.e., $j = 2n$. By symmetry, the computations for the first column are equivalent. From (64) we obtain

$$\det \mathbf{M}^{(2n,i,2n)} = \left(\prod_{j=\lfloor \frac{i+2}{2} \rfloor}^{n} \sqrt{1-q_j^2}\right) \left(\prod_{j=\lfloor \frac{i+1}{2} \rfloor}^{n-1} \left(-\frac{1}{\sqrt{\sigma_j}}\right)\right) \det \mathbf{M}^{(i-1)}. \tag{25}$$

The following corollary is a direct consequence of Lemma 7 and (25). It allows us to write the entries of the Green's function $\mathbf{M}_{\text{Green}}$ in terms of the sequence (Q_j), leading to a stability estimate depending on (Q_j) (Theorem 10).

Corollary 8 *The entries of the last column of $\mathbf{M}_{\text{Green}}^{(2n)}$ can be written in the form*

$$\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{i,2n} = (-1)^{i+1} \left(\prod_{\ell=\lfloor \frac{i+2}{2} \rfloor}^{n} \frac{\sqrt{1-q_\ell^2}}{(1+q_\ell Q_{\ell-1})\sqrt{\sigma_{\ell-1}}}\right) \times \begin{cases} \sigma_{\frac{i}{2}} Q_{\frac{i}{2}} & i \text{ is even,} \\ \sqrt{\sigma_{\frac{i-1}{2}}} & i \text{ is odd.} \end{cases} \tag{26}$$

For the modulus of the entries of the odd rows it holds

$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n-2m+1,2n}\right| = \left|\prod_{\ell=n-m+1}^{n} \frac{\sqrt{1-q_\ell^2}}{(1+q_\ell Q_{\ell-1})}\right|, \qquad 1 \leq m \leq n, \tag{27}$$

and for the even rows:

$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n,2n}\right| = |Q_n|,$$
$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n-2m,2n}\right| = \left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{2n-2m+1,2n}\right| |Q_{n-m}|, \qquad 1 \leq m \leq n-1.$$

The estimates on the coefficients in $\mathbf{M}_{\text{Green}}^{(2n)}$ is based on the representation (27), i.e., we have to estimate the expression

$$G_{n,m} := \left|\prod_{\ell=n-m+1}^{n} \frac{\sqrt{1-q_\ell^2}}{(1+q_\ell Q_{\ell-1})}\right|, \qquad 1 \leq m \leq n, \tag{28}$$

with Q_j as in (23a).

Lemma 9 *(i) Let $q_j \in [-q, q]$ for all $1 \leq j \leq n$ and some $0 < q < 1$. Then*

$$G_{n,m} \leq \frac{1}{\sqrt{1-|Q_{n-m}|^2}}, \qquad 1 \leq m \leq n.$$

(ii) If additionally $Q_{n-1} = -q_n$ holds, we have

$$G_{n,m} = \frac{1}{\sqrt{1 - |Q_{n-m}|^2}}, \qquad 1 \le m \le n. \tag{29}$$

Proof.

(i) We prove the statement by induction on m. For $m = 1$, we employ simple calculus and obtain

$$G_{n,1} \le \max_{q_n \in [-q,q]} \left| \frac{\sqrt{1-q_n^2}}{1 + q_n Q_{n-1}} \right| = \max_{x \in [0,q]} \frac{\sqrt{1-x^2}}{1 - x\,|Q_{n-1}|} \le \frac{1}{\sqrt{1 - |Q_{n-1}|^2}}.$$

Next we assume that the estimate holds for $G_{n,\ell}$, $1 \le \ell \le m-1$. For m, we get

$$\begin{aligned} G_{n,m} &= \frac{\sqrt{1 - q_{n-m+1}^2}}{|1 + q_{n-m+1} Q_{n-m}|} \prod_{\ell=n+2-m}^{n} \frac{\sqrt{1-q_\ell^2}}{|1 + q_\ell Q_{\ell-1}|} \\ &\le \frac{\sqrt{1 - q_{n-m+1}^2}}{|1 + q_{n-m+1} Q_{n-m}|} \frac{1}{\sqrt{1 - |Q_{n-m+1}|^2}} \\ &= \frac{\sqrt{1 - q_{n-m+1}^2}}{\sqrt{|1 + q_{n-m+1} Q_{n-m}|^2 - |q_{n-m+1} + Q_{n-m}|^2}} \\ &= \frac{1}{\sqrt{1 - |Q_{n-m}|^2}}. \end{aligned} \tag{30}$$

(ii) Since $Q_{n-1} = -q_n$, we can compute $G_{n,1}$ directly

$$G_{n,1} = \left| \frac{\sqrt{1-q_n^2}}{1 + q_n Q_{n-1}} \right| = \frac{1}{\sqrt{1 - |Q_{n-1}|^2}}.$$

The induction step follows analogously to (30).

■

With the estimate of the entries of $\mathbf{M}_{\text{Green}}^{(2n)}$ of Lemma 9, we can state the following stability result.

Theorem 10 *The solution u of (13) with piecewise constant wave speed c as in* (12) *satisfies*

$$\left\| u^{(k)} \right\|_{L^2(\Omega)} \le 4 \frac{c_{\max}}{c_{\min}^k} \omega^{k-1} \max\{|g_1|, |g_2|\} \max_{1 \le j \le n} \frac{1}{\sqrt{1 - |Q_j|^2}} \tag{31}$$

for $k = 0, 1, 2$.

Remark 11 *If c is constant, the qualitative frequency dependence in the stability estimate* (31) *coincides with known results (cf. [20, 9]).*

Remark 12 *The final result (Theorem 22) is proved by a further analysis of the sequence* (Q_j) *or, more precisely, of the distance* $1-|Q_j|^2$. *We emphasize that the magnitude of the last term in* (31) *does not necessarily increase in* ω, *but depends on the interplay of* h_j, c_j *and* ω *via the phase factor* σ_j *defined in* (14)*(resonance effect)* .

Proof. From (16) we obtain for any $k=0,1,2$

$$\left\|u^{(k)}\right\|_{L^2(\Omega)} \leq \sqrt{2} \max_{1\leq j\leq n+1} \left(\frac{\omega}{c_j}\right)^k (|A_j|+|B_j|) .$$

The coefficients A_j, B_j are contained in the solution vector $\mathbf{x}^{(2n)}$ and can be written in the form (cf. (22))

$$\begin{aligned}\mathbf{x}_j^{(2n)} &= \left(\left(\mathbf{D}^{(2n)}\right) \mathbf{M}_{\text{Green}}^{(2n)} \left(\mathbf{D}^{(2n)}\right) \mathbf{r}^{(2n)}\right)_j \\ &= \frac{\mathrm{i}}{2\omega}\left(\alpha_{1,1}\sqrt{c_1}\,\mathrm{e}^{\mathrm{i}\,\frac{\omega}{c_1}}\, g_1 \mathbf{d}_j^{(2n)} \left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{j,1} + \frac{\sqrt{c_{n+1}}}{\alpha_{n+1,n}}\,\mathrm{e}^{\mathrm{i}\,\frac{\omega}{c_{n+1}}}\, g_2 \mathbf{d}_j^{(2n)} \left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{j,2n}\right)\end{aligned}$$

with (cf. (20))

$$\mathbf{d}^{(2n)} := \left(\alpha_{1,1}\sqrt{c_1}, \frac{\sqrt{c_2}}{\alpha_{2,1}}, \alpha_{2,2}\sqrt{c_2}, \frac{\sqrt{c_3}}{\alpha_{3,2}}, \ldots, \alpha_{n,n}\sqrt{c_n}, \frac{\sqrt{c_{n+1}}}{\alpha_{n+1,n}}\right)^{\intercal}.$$

Hence,

$$\max_{1\leq j\leq n} \max\{|A_j|, |B_j|\} \leq \frac{c_{\max}}{\omega} \max\{|g_1|, |g_2|\} \max_{1\leq i\leq 2n} \max_{j\in\{1,2n\}} \left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{i,j}\right|. \tag{32}$$

■

Lemma 13 *A lower bound for the norm of* $u^{(k)}$, $k=0,1,2$, *is given by*

$$\left\|u^{(k)}\right\|_{L^2(\Omega)} \geq \sqrt{\frac{2}{15}h_j} \left(\frac{\omega}{c_j}\right)^k \frac{\frac{\omega}{c_j}h_j}{1+\frac{\omega}{c_j}h_j} \max\{|A_j|, |B_j|\},$$

where u *and* A_j, B_j *are related through* (16) *and (18).*

Proof. It holds

$$\left\|u^{(k)}\right\|_{L^2(\Omega)}^2 \geq \left\|u^{(k)}\right\|_{L^2(\tau_j)}^2 = \left\langle \begin{pmatrix} A_j \\ B_j \end{pmatrix}, E^{(k)} \begin{pmatrix} A_j \\ B_j \end{pmatrix} \right\rangle$$

with the Euclidean scalar product $\langle\cdot,\cdot\rangle$, the Hermitian matrix

$$E_{n,m}^{(k)} := \left(\frac{\omega}{c_j}\right)^{2k} h_j \begin{bmatrix} 1 & (-1)^k\,\mathrm{e}^{\mathrm{i}\,\frac{\omega}{c_j}(2x_{j-1}+h_j)} \operatorname{sinc}\left(\frac{\omega}{c_j}h_j\right) \\ (-1)^k\,\mathrm{e}^{-\mathrm{i}\,\frac{\omega}{c_j}(2x_{j-1}+h_j)} \operatorname{sinc}\left(\frac{\omega}{c_j}h_j\right) & 1 \end{bmatrix},$$

and the convention for the sinc function $\operatorname{sinc}(x) := (\sin x)/x$ for $x\neq 0$ and $\operatorname{sinc}(0) := 1$. The eigenvalues of $E^{(k)}$ are given by

$$\lambda_{1,2}^{(k)} = \left(\frac{\omega}{c_j}\right)^{2k} h_j \left(1 \pm \operatorname{sinc}\left(\frac{\omega}{c_j}h_j\right)\right).$$

Simple calculus leads to

$$\left|\lambda_{1,2}^{(k)}\right| \geq \left(\frac{\omega}{c_j}\right)^{2k} h_j \times \left\{ \begin{array}{cc} \frac{2}{15}\left(\frac{\omega}{c_j}h_j\right)^2 & \frac{\omega}{c_j}h_j \leq 2 \\ \frac{1}{2} & \frac{\omega}{c_j}h_j \geq 2 \end{array} \right\} \geq \frac{2}{15} h_j \left(\frac{\omega}{c_j}\right)^{2k} \frac{\left(\frac{\omega}{c_j}h_j\right)^2}{1+\left(\frac{\omega}{c_j}h_j\right)^2}$$

and some straightforward manipulations result in the asserted estimate. ■

4 Construction of Configurations with "good" and "bad" Stability Properties

We construct two slightly different configurations of c_j and h_j, to find two very different behaviour of the growth of $|Q_j|$ leading to qualitatively different maximal entries of $\mathbf{M}_{\text{Green}}^{(2n)}$. Figure 1 shows the corresponding solution for a specific example in both cases. In both cases, we first choose the *mathematical parameters* σ_j, q_j such that the sequence $(Q_\ell)_\ell$ exhibits the desired behaviour and then determine the corresponding physical parameters.

4.1 Well-behaved Case

Given an arbitrary frequency ω, we construct an example of a highly oscillating configuration, where $|Q_j|$ can be bounded away from 1 independently of ω. Recalling the definition of Q_j, we write Q_2 as

$$Q_2 = \frac{q_2 + \frac{q_1}{\sigma_1}}{\sigma_2\left(1+\frac{q_1 q_2}{\sigma_1}\right)}.$$

We see that the modulus of Q_2 is minimized, if q_1, q_2 and σ_1 are chosen such that $q_2 + \frac{q_1}{\sigma_1} = 0$. This is true, for example, if $q_1 = -q_2$ and $\sigma_1 = 1$. If indeed $Q_2 = 0$, the same choice for q_3, q_4 and σ_3 leads to $Q_4 = 0$. As a result of this observation, we define the oscillating, piecewise constant wave speed by

$$c_j = \begin{cases} c\,(1-q) & j \text{ odd}, \\ c\,(1+q) & j \text{ even}, \end{cases} \qquad 1 \leq j \leq n+1, \tag{33}$$

for some $c > 0$ and $q \in (0,1)$, so that $q = \frac{c_{\max}-c_{\min}}{c_{\max}+c_{\min}}$. Indeed, we then obtain $q_j = (-1)^{j+1} q$. In order to have $\sigma_j = 1$ for all j (cf. (14)), we consider the case $\frac{\omega h_j}{c_j} \gtrsim 1$ and step widths h_j which satisfy

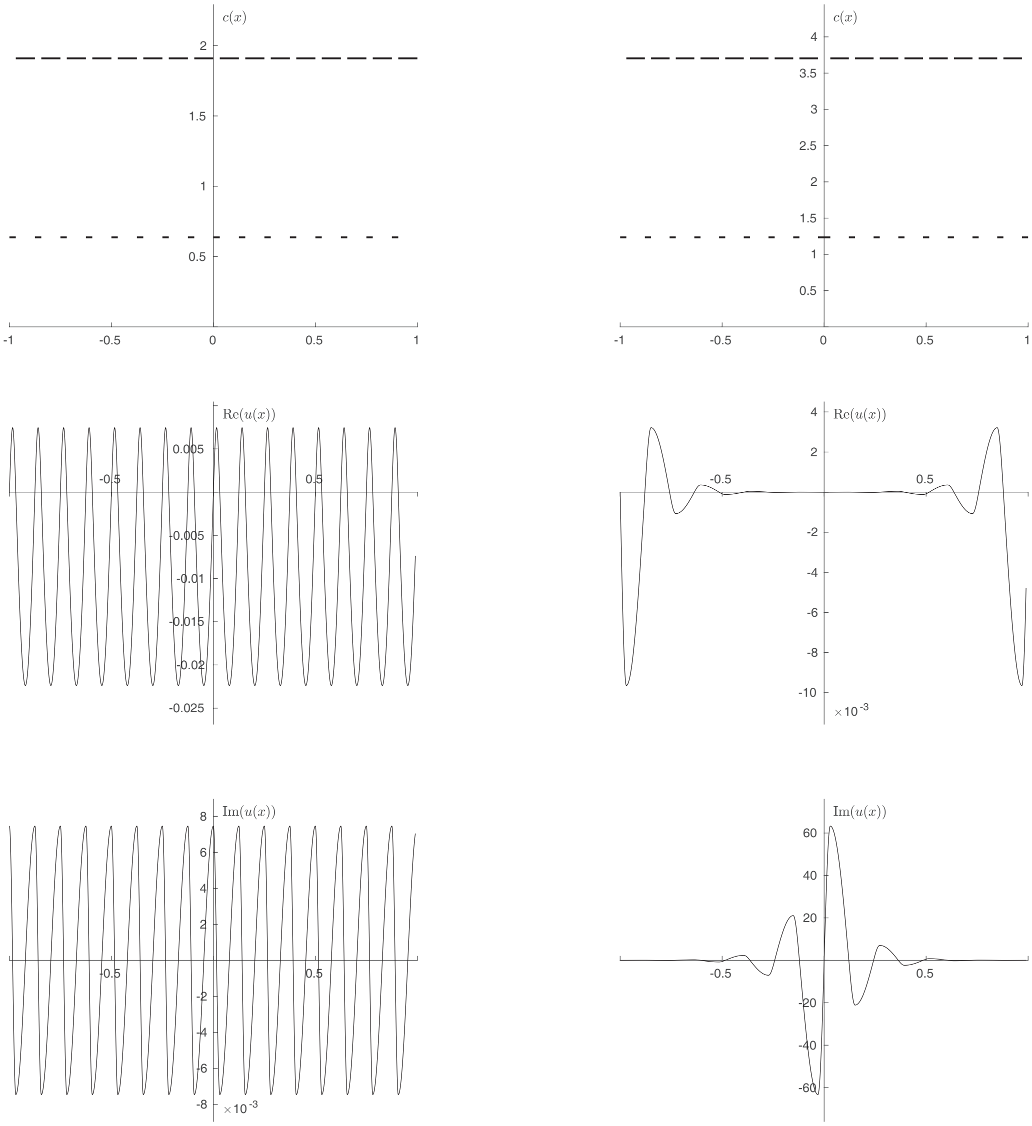
$$\frac{\omega h_j}{c_j \pi} \in \mathbb{N} \text{ for all } j. \tag{34}$$

To be concrete, we choose n odd and

$$h_j := c_j \frac{\pi}{\omega}$$

The side condition

$$\sum_{j=1}^{n+1} h_j = \sum_{j=1}^{n+1} c_j \frac{\pi}{\omega} = 2$$



(a) Solution u, with $n = 32$. The stability constant is bounded from above independently of ω

(b) Solution u, with $n = 33$. The stability constant grows exponentially in ω.

Figure 1: Examples for a piecewise constant wave speed and corresponding solution of the Helmholtz problem. In both examples we set $\omega = 64$, $q = 0.5$, $g_1 = 0$, $g_2 = 1$, h_i, c as in §4.1 and §4.2, respectively.

leads to

$$c = \frac{2}{n+1}\frac{\omega}{\pi}, \quad h_j = \frac{2}{n+1} \times \begin{cases} 1-q & j \text{ odd}, \\ 1+q & j \text{ even}, \end{cases}$$

and

$$x_j = \begin{cases} -1 + \frac{2(j-q)}{n+1} & j \text{ odd}, \\ -1 + \frac{2j}{n+1} & j \text{ even}, \end{cases} \qquad 1 \le j \le n+1.$$

In any case, condition (34) implies

$$Q_j = \begin{cases} q & j \text{ odd}, \\ 0 & j \text{ even}, \end{cases}$$

and

$$\frac{\sqrt{1-q_j^2}}{1+q_j Q_{j-1}} = \begin{cases} \frac{1}{\sqrt{1-q^2}} & j \text{ even}, \\ \sqrt{1-q^2} & j \text{ odd}. \end{cases}$$

Hence, the product of two subsequent factors in $G_{n,m}$ equals 1, so that

$$G_{n,m} \le \frac{1}{\sqrt{1-q^2}}.$$

Note that

$$\frac{1}{\sqrt{1-q^2}} = \frac{1}{2}\left(\sqrt{\frac{c_{\max}}{c_{\min}}} + \sqrt{\frac{c_{\min}}{c_{\max}}}\right) \le \sqrt{\frac{c_{\max}}{c_{\min}}}$$

From Corollary 8, we conclude that

$$\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{j,2n}\right| \le \sqrt{\frac{c_{\max}}{c_{\min}}}.$$

By employing the same arguments to the first column of $\mathbf{M}_{\text{Green}}^{(2n)}$, we get $\left|\left(\mathbf{M}_{\text{Green}}^{(2n)}\right)_{j,1}\right| \le \sqrt{\frac{c_{\max}}{c_{\min}}}$. Hence, Theorem 10 implies

$$\left\|u^{(k)}\right\|_{L^2(\Omega)} \le \frac{c_{\max}^{3/2}}{c_{\min}^{k+1/2}}\omega^{k-1}\max\left\{|g_1|, |g_2|\right\}.$$

This shows that the known bounds in [8] and [12], which grow exponentially in the number of jumps, are very pessimistic for this example. We summarise the findings of this example. Let $\omega > 0$ be given and let $n = O(\omega)$ be odd. Choose some $q \in [0,1[$. Define

$$c := \frac{2}{n+1}\frac{\omega}{\pi} = O(1), \quad h_j = \frac{2}{n+1} \times \begin{cases} 1-q & j \text{ odd}, \\ 1+q & j \text{ even}, \end{cases},$$

and the piecewise constant wave speed by

$$c_j = \begin{cases} c(1-q) & j \text{ odd}, \\ c(1+q) & j \text{ even}. \end{cases}$$

The stability constant for problem (13) is bounded independent of ω as in the case of constant wave speed.

4.2 Critical Case

For a given frequency ω, we construct a class of configurations of c_j and h_j which are in resonance, i.e. where we observe a stability constant which grows exponentially in the frequency. However, we remark that these configurations seem to be very rare, since they depend on a specific relation between c_j, h_j and ω in a very sensitive way. Recall the notation as in (23) and $G_{n,m}$ is defined as in (28). If $Q_{n-1} = -q_n$ we know from Lemma 9 b) that

$$G_{n,\ell} = \frac{1}{\sqrt{1 - |Q_{n-\ell}|^2}}, \qquad 1 \leq \ell \leq n. \tag{35}$$

We investigate the question whether some $|Q_k|^2$ can become exponentially close to 1 with respect to growing k. Let $n = 2k$ be even. We will choose the first $k-1$ entries of $(\sigma_j)_{j=1}^n$ such that $|Q_j|$ increases, and the last entries will be chosen such that $|Q_j|$ decreases until $Q_{n-1} = -q_n$. We employ the same ansatz for the perfectly oscillating wave speed c as in (33), so that $q = \frac{c_{\max} - c_{\min}}{c_{\max} + c_{\min}} \in (0,1)$ and $q_j = (-1)^{j+1} q$. We choose the phase factors $\sigma_j = \exp\left(-2\,\mathrm{i}\,\frac{\omega h_{j+1}}{c_{j+1}}\right)$, and will adjust the value of the constant c and $(h_j)_{j=1}^{n+1}$. We choose the first $k-1$ phase factors σ_j according to

$$\sigma_j := \operatorname{sign}(q_j q_{j+1}) = -1, \qquad \forall 1 \leq j \leq k-1.$$

In Lemma 15, we will prove that then

$$Q_j = (-1)^j \frac{(1+q)^j - (1-q)^j}{(1+q)^j + (1-q)^j}, \qquad 1 \leq j \leq k-1.$$

and

$$Q_k = \frac{(-1)^{k-1}}{\sigma_k} \frac{(1+q)^k - (1-q)^k}{(1+q)^k + (1-q)^k}.$$

Now, setting $\sigma_k = 1$ yields

$$\begin{aligned} Q_{k+1} &= \frac{1}{\sigma_{k+1}} \frac{q_{k+1} + Q_k}{1 + q_{k+1} Q_k} = \frac{(-1)^{k-1}}{\sigma_{k+1}} \frac{-q + |Q_k|}{1 - q|Q_k|} \\ &= \frac{(-1)^{k-1}}{\sigma_{k+1}} \frac{(1+q)^{k-1} - (1-q)^{k-1}}{(1+q)^{k-1} + (1-q)^{k-1}}. \end{aligned}$$

With the choice $\sigma_j = -1$, for all $k+1 \leq j \leq n$, and proceeding in an analogue way we obtain for $1 \leq j \leq k$

$$Q_{k+j} = (-1)^{k-j+1} \frac{(1+q)^{k-j} - (1-q)^{k-j}}{(1+q)^{k-j} + (1-q)^{k-j}}.$$

This leads finally to (note that $q_n = -q$ for even n)

$$Q_{n-1} = q.$$

Hence,

$$|Q_k| = \frac{(1+q)^k - (1-q)^k}{(1+q)^k + (1-q)^k}$$

is exponentially close to 1 with respect to $k \to \infty$ and the corresponding element in the Green's function $\mathbf{M}_{\text{Green}}$ tends exponentially towards infinity (cf. (29), (27)). For $n = 2k$, $m = k$ and our choices of q_ℓ and σ_ℓ, it holds

$$\left| \left(\mathbf{M}_{\text{Green}}^{(2n)} \right)_{2k+1,2n} \right| = \frac{1}{\sqrt{1 - |Q_k|^2}} = \frac{1}{2} \left(\left(\frac{1+q}{1-q} \right)^{k/2} + \left(\frac{1-q}{1+q} \right)^{k/2} \right), \tag{36}$$

i.e., this matrix entry grows exponentially with respect to increasing k. By adjusting the mesh widths h_j and the constant c such that all $\exp\left(-2\frac{\mathrm{i}\,\omega h_{j+1}}{c_{j+1}}\right)$ coincide with the phase factors σ_j as defined in this example, we have a configuration, where the stability constant grows exponentially in the number of jumps. In order to achieve

$$\sigma_j = \begin{cases} 1 & j = k, \\ -1 & \text{otherwise}, \end{cases}$$

we obtain the following conditions for the mesh sizes

$$h_j = \frac{\pi}{\omega} \times \begin{cases} c_j m_j & j = k+1, \\ \\ c_j \left(m_j + \frac{1}{2} \right) & \text{otherwise}, \end{cases} \tag{37}$$

for any sequence $m_j \in \mathbb{Z}$. Moreover, the side condition

$$\sum_{j=1}^{n+1} h_j = 2 \tag{38}$$

applies. In (37), we choose $m_j = 1$ for $j = k+1$ and $m_j = 0$ otherwise. Hence (38) is equivalent to

$$2c_{k+1} + \sum_{\substack{i=1 \\ i \neq k+1}}^{2k+1} c_i = \frac{4\omega}{\pi}$$

Let k be even. The definition (33) implies

$$c = \frac{2\omega}{\pi\,(1 - q + k)}.$$

We summarise the findings of this example. Let $\omega > 0$ be given and let $n = 2k$ for even $k \in \mathbb{N}$ and $k = O(\omega)$. Choose some $q \in [0, 1[$. Define

$$c := \frac{2\omega}{\pi\,(1 - q + k)} = O(1)$$

and the piecewise constant wave speed by

$$c_j = \begin{cases} c\,(1 - q) & j \text{ odd}, \\ c\,(1 + q) & j \text{ even}. \end{cases}$$

Define mesh sizes according to

$$h_j = \frac{c_j}{\omega}\pi \times \begin{cases} 1 & j = k+1, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and mesh points x_j recursively by $x_0 := -1$ and for $1 \leq j \leq n+1$ by $x_j = x_{j-1} + h_j$. The stability constant for problem (13) grows exponentially with increasing ω.

Remark 14 *From Lemma 13 we conclude that the norm $\|u^{(k)}\|$ may grow exponentially in ω; indeed, for this example, the choice $g_1 = 0$, $g_2 = 1$ and $\omega = k$ for $k \geq 2$ leads to*

$$\frac{4}{3\pi} \leq c \leq \frac{2}{\pi}, \quad \frac{4}{3\pi}(1-q) \leq c_j \leq \frac{4}{\pi}, \quad \frac{2}{3}\frac{(1-q)}{\omega} \leq h_j \leq \frac{4}{\omega}, \quad \frac{\pi}{2} \leq \frac{\omega h_j}{c_j} \leq \pi.$$

Thus Lemma 13 implies

$$\left\| u^{(k)} \right\|_{L^2(\Omega)} \geq \frac{2\pi\sqrt{1-q}}{3\sqrt{5}(\pi+2)} \left(\frac{\pi}{4}\right)^k \omega^{k-1/2} \max\{|A_j|, |B_j|\}.$$

From (36) and the proof of Theorem 10 we conclude that

$$\max_j \max\{|A_j|, |B_j|\} \geq \left(\frac{1+q}{1-q}\right)^{\omega/2} \frac{1-q}{3\pi\omega}$$

holds. The combination of these estimates leads to

$$\begin{aligned} \left(\int_\Omega |u'|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2\right)^{\frac{1}{2}} &\geq \frac{\pi(1-q)^{3/2}}{9\sqrt{5}(\pi+2)} \left(\frac{1}{2} + \frac{\pi}{\sqrt{5(1-q)}(\pi+2)}\right) \omega^{-1/2} \left(\frac{1+q}{1-q}\right)^{\omega/2} \\ &\geq C_q \alpha_q^\omega \end{aligned}$$

for some $C_q > 0$ and $\alpha_q \in (0,1)$ depending only on $q \in (0,1)$.

5 Recursive Representation of the Inverse Green's Function

In Section 5.1, we discuss the possible growth of $|Q_j|$ for fixed ω and general parameters c_j and h_j. This will directly lead to a stability estimate if the step sizes are above resonance, i.e. where $\frac{h}{c} \geq O\left(\frac{1}{\omega}\right)$ (cf. Section 5.2). In Section 5.3, we will restrict to the case where c oscillates perfectly between two values and we will show, that for any general configuration of $(h_j)_{j=1}^{n+1}$, the stability constant cannot exceed the exponential growth with respect to ω described in Section 4.2.

5.1 The Influence of the Phase Factors σ_i

For the estimate of Q_j, we have first to introduce some quantities and conventions. The coefficient Q_j depends on $(q_i)_{i=1}^j$ and $(\sigma_i)_{i=1}^j$. The index j in $Q_j(\mathbf{q}, \boldsymbol{\sigma})$ always indicates the lengths of $\mathbf{q} = (q_i)_{i=1}^j$ and $\boldsymbol{\sigma} = (\sigma_i)_{i=1}^j$, where $q_i \in [-q, q]$ and $\sigma_i \in \mathcal{C}$ for all $1 \leq i \leq j$. For given $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, let $\widehat{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{q}, \boldsymbol{\sigma}) = (\hat{\sigma}_1, \ldots, \hat{\sigma}_{j-1}, \hat{\sigma}_j)^\intercal$ be defined by

$$\begin{aligned} &\hat{\sigma}_i := \operatorname{sign}(q_i q_{i+1}) \in \{-1, 1\} \qquad \forall 1 \leq i \leq j-1, \\ &\hat{\sigma}_j := \sigma_j. \end{aligned} \tag{39}$$

In view of Lemma 9, we investigate $|Q_{n-m}|$ for $m = 1, 2, \ldots, n$.

Lemma 15 *Recall the definitions of Q_j and Q_j' as in (23).*

(i) It holds
$$\max_{\substack{\sigma_i\in\mathcal{C}\\1\leq i\leq j}} |Q_j(\mathbf{q},\boldsymbol{\sigma})| = |Q_j(\mathbf{q},\widehat{\boldsymbol{\sigma}})|, \tag{40}$$
i.e., the maximizer $(\hat{\sigma}_i)_{i=1}^{j-1}$ coincides with the maximizer for a larger $j' > j$ in the first $j-1$ components.

(ii) The quantity $Q_j'(\mathbf{q}) := \sigma_j Q_j(\mathbf{q},\widehat{\boldsymbol{\sigma}})$ satisfies $Q_j'(\mathbf{q}) \in \mathbb{R}$, $\left|Q_j'(\mathbf{q})\right| < 1$ and
$$\operatorname{sign}\left(Q_j'(\mathbf{q})\right) = \operatorname{sign}(q_j). \tag{41}$$
For given $\mathbf{q}\in[-q,q]^j$ and $\widehat{\boldsymbol{\sigma}} = (\hat{\sigma}_i)_{i=1}^j$ as in (39) it holds
$$\operatorname{sign}(Q_\ell(\mathbf{q},\widehat{\boldsymbol{\sigma}})) = \operatorname{sign}(q_{\ell+1}) \qquad \forall 1\leq\ell\leq j-1. \tag{42}$$

(iii) For any sequence $\tilde{\mathbf{q}} = (\tilde{q}_i)_{i=1}^j$ such that $\tilde{q}_i \in \{-q,q\}$ and corresponding $\hat{\boldsymbol{\sigma}}$, the maximum over all $q_i\in[-q,q]$ is given by
$$\begin{aligned}\max_{\mathbf{q}\in[-q,q]^j}\max_{\substack{\sigma_i\in\mathcal{C}\\1\leq i\leq j}} |Q_j(\mathbf{q},\boldsymbol{\sigma})| &= |Q_j(\tilde{\mathbf{q}},\widehat{\boldsymbol{\sigma}}(\tilde{\mathbf{q}}))| \\ &= \frac{(1+q)^j-(1-q)^j}{(1+q)^j+(1-q)^j}.\end{aligned} \tag{43}$$

Proof. We prove the equalities in (40) and (41) by induction. Both equalities are trivial for $j=1$. For $j=2$, we get
$$\max_{\sigma_1,\sigma_2\in\mathcal{C}} |Q_2(\mathbf{q},\boldsymbol{\sigma})| = \max_{\sigma_1\in\mathcal{C}}\left|\frac{q_2+\frac{q_1}{\sigma_1}}{1+q_2\frac{q_1}{\sigma_1}}\right| = \max_{\sigma_1\in\mathcal{C}}\sqrt{\frac{q_2^2+q_1^2+2q_1q_2\operatorname{Re}\sigma_1}{1+q_2^2q_1^2+2q_1q_2\operatorname{Re}\sigma_1}} = \frac{|q_1|+|q_2|}{1+|q_1q_2|}$$
and the maximum is achieved for $\sigma_1 := \hat{\sigma}_1$ with $\hat{\sigma}_1 := \operatorname{sign}(q_1q_2)$. In this case
$$\operatorname{sign} Q_2'(\mathbf{q}) = \operatorname{sign}\frac{q_2+\frac{q_1}{(\operatorname{sign} q_1q_2)}}{1+\frac{q_2q_1}{(\operatorname{sign} q_1q_2)}} = \operatorname{sign} q_2.$$
Next we consider the case $j\geq 3$. By induction we assume that (40) and (41) hold for $j' = 1,2,\ldots,j-1$. Thus, we get by the maximum modulus principle
$$\max_{\substack{\sigma_i\in\mathcal{C}\\1\leq i\leq j}} |Q_j(\mathbf{q},\boldsymbol{\sigma})| = \max_{\sigma_{j-1}\in\mathcal{C}}\left|\frac{q_j+\frac{Q_{j-1}'(\mathbf{q})}{\sigma_{j-1}}}{1+q_j\frac{Q_{j-1}'(\mathbf{q})}{\sigma_{j-1}}}\right|. \tag{44a}$$
Similar as before we see that the right-hand side attains its maximum for $\sigma_{j-1} = \hat{\sigma}_{j-1}$ with $\hat{\sigma}_{j-1} = \operatorname{sign}\left(q_jQ_{j-1}'(\mathbf{q})\right)$, i.e.,
$$\max_{\substack{\sigma_i\in\mathcal{C}\\1\leq i\leq j}} |Q_j| = \frac{|q_{j+1}|+\left|Q_{j-1}'(\mathbf{q})\right|}{1+|q_{j+1}|\left|Q_{j-1}'(\mathbf{q})\right|}. \tag{44b}$$

By induction we have $\operatorname{sign} Q'_{j-1}(\mathbf{q}) = \operatorname{sign} q_j$ so that $\hat{\sigma}_j = \operatorname{sign}(q_j q_{j+1})$. Thus,

$$\begin{aligned}
\operatorname{sign}\left(Q'_j(\mathbf{q},\widehat{\boldsymbol{\sigma}})\right) &= \operatorname{sign}\left(\frac{q_j + \frac{Q'_{j-1}(\mathbf{q})}{\operatorname{sign}(q_{j-1}q_j)}}{1 + q_j \frac{Q'_{j-1}(\mathbf{q})}{\hat{\sigma}_{j-1}}}\right) = \operatorname{sign}\left(q_j + (\operatorname{sign} q_j)\left|Q'_{j-1}(\mathbf{q})\right|\right) \\
&= (\operatorname{sign} q_j)\operatorname{sign}\left(|q_j| + \left|Q'_{j-1}(\mathbf{q})\right|\right) = \operatorname{sign} q_j.
\end{aligned}$$

The sign of Q_j as in (42) can be determined by

$$\operatorname{sign}\left(Q_j(\mathbf{q},\widehat{\boldsymbol{\sigma}})\right) = \operatorname{sign}\left(\hat{\sigma}_j Q'_j\right) = \left(\operatorname{sign} Q'_j\right)\operatorname{sign}(q_j q_{j+1}) = \operatorname{sign} q_{j+1}.$$

Hence part (i) and (ii) are proved.

To prove the bound (43) we observe that the coefficients $\rho_j := |Q_j|$ are majorized by the sequence

$$\tilde{r}_1(q) = q \quad \text{and } \forall j \geq 2 \quad \tilde{r}_j := \frac{q + \tilde{r}_{j-1}}{1 + q\tilde{r}_{j-1}}.$$

The closed form of this recursion is given by

$$\tilde{r}_j(q) = \frac{(1+q)^j - (1-q)^j}{(1+q)^j + (1-q)^j}. \tag{45}$$

■

Remark 16 *The combination of (31) and Lemma 15(iii) implies*

$$\begin{aligned}
\left(\int_\Omega |u'|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2\right)^{\frac{1}{2}} &\leq 8\frac{c_{\max}}{c_{\min}} \max\{|g_1|, |g_2|\} \max_{1\leq j\leq n} \frac{1}{\sqrt{1 - |Q_j|^2}} \\
&\leq 4\frac{c_{\max}}{c_{\min}}\left(\kappa^{n/2} + \kappa^{-n/2}\right)\max\{|g_1|, |g_2|\}
\end{aligned}$$

with the condition number κ *of the wave speed as in (6). This shows that for* fixed *number of jumps the stability estimate is independent of the wave number ω. Such types of estimates are also proved in [8], [12].*

From now on, we will consider c piecewise constant and perfectly oscillating between two values $c_{\min}$ and $c_{\max}$, i.e.

$$c_j = \begin{cases} c_0 & \text{if } j \text{ is odd}, \\ c_1 & \text{if } j \text{ is even}, \end{cases}$$

with $0 < c_{\min} = \min\{c_0, c_1\} \leq \max\{c_0, c_1\} = c_{\max} < \infty$. In that case, we know from Lemma 15 that $|Q_j|$ increases maximally for a certain specific choice of $(\sigma_i)_i$. However, motivated by the example in Section 4.1, we know that this can be very pessimistic. In particular if $h \leq \frac{\delta}{\omega}$ for sufficiently small δ, then $\sigma_{i-1} = \exp\left(-2\,\mathrm{i}\,\frac{h_i\omega}{c_i}\right) \approx 1$. If $\sigma_i = 1$ for all i (and $q_i = (-1)^{i+1}q$), we have seen that $(Q_i)_i$ can be bounded away from 1 independent of the number of jumps (cf. Section 4.1). The idea we follow is to split the domain into two types of subsequences of h_i. The first type covers parts of the domain where $\omega\frac{h_i}{c_i}$ is bounded from below. In this case, we can be bound

$|Q_m|$ from above with respect to the frequency ω using estimates of the type (45). For parts of the domain where $\omega \frac{h_i}{c_i}$ is small, we will use another approach, but with a similar result. In the end, the two estimates can be combined by finding an upper bound on $|Q_{j+m}|$ with respect to the value $|Q_j|$ instead of $|Q_0|$. The following corollary restates Lemma 15 for this purpose and the proof is a repetition of arguments.

Corollary 17 *Define the sequence*

$$Q_j = \frac{q_j + Q_{j-1}}{\sigma_j(1 + q_j Q_{j-1})}$$

for $Q_1 = \frac{\tilde{q}}{\sigma_1}$ *for* $0 \le \tilde{q} < 1$*. Also assume that* $q_j = (-1)^{j+1} q$ *for* $2 \le j \le n-1$ *and some* $0 < q < 1$*. We define*

$$r_{\tilde{q},j}(q) := \frac{(1+\tilde{q})(1+q)^{j-1} - (1-\tilde{q})(1-q)^{j-1}}{(1+\tilde{q})(1+q)^{j-1} + (1-\tilde{q})(1-q)^{j-1}}.$$

(i) $r_{\tilde{q},j}$ *is increasing in* $\tilde{q}$*.*

(ii) If $\sigma_j = -1$ *for all* $1 \le j \le n$*, then*

$$|Q_j| = r_{\tilde{q},j}(q)$$

and $\operatorname{sign}(Q_j) = (-1)^{j+1}$ *for all* $1 \le j \le n$*.*

(iii) If $(\sigma_j)_{j=1}^{\infty} \subset \mathcal{C}$ *is a general sequence, then*

$$|Q_{m+j}| \le r_{|Q_j|,m}(q), \quad \forall\ 1 \le m + j \le n.$$

5.2 Estimate of Q_m for Step Sizes above Resonance Case

Assume that $\omega \frac{h_j}{c_j} > \varepsilon$ for some $\varepsilon > 0$ and for all $1 \le j \le n+1$. Then we have

$$(n+1) \cdot \varepsilon \le \sum_{j=1}^{n+1} \omega \frac{h_j}{c_j} \le 2 \frac{\omega}{c_{\min}},$$

and therefore the length of the sequence $(Q_j)_{j=1}^n$ is bounded by $n \le \frac{2\omega}{\varepsilon c_{\min}} - 1 = N(\varepsilon, \omega)$. Using Lemma 15 and Corollary 17, we derive

$$|Q_j| \le r_{|Q_0|, N(\varepsilon,\omega)}(q), \quad 1 \le j \le n. \tag{46}$$

The result obtained so far is summarised in the next theorem.

Proposition 18 *Assume that the step size* h_i *is bounded from below, i.e. the number of jumps is bounded from above by* $\alpha\omega$ *for some* α*, and* c *is piecewise constant. Then the stability constant in (10) of the Helmholtz problem* (13) *satisfies*

$$C_{\text{stab}} \lesssim \alpha_q^{-\frac{\omega}{c_{\min}}}, \quad \text{for some } 0 < \alpha_q < 1.$$

This result coincides with the stability estimate proved in [8] (for slightly different boundary conditions) and shows that our theory recovers this result.

5.3 Estimate of Q_m for small Step Sizes

In this section, we discuss the growth of the magnitude of Q_j for small widths h_j, when the wave speed c is perfectly oscillating between two values. In this case, we cannot control the number of jumps, however we know that the sum of the widths equals the interval length $|\Omega| = 2$, i.e.

$$\sum_{j=1}^{n+1} \omega \frac{h_j}{c_j} \leq \frac{\omega}{c_{\min}} \sum_{j=1}^{n+1} h_j \leq \frac{2\omega}{c_{\min}}$$

and this will play a key role in the proof of Proposition 19.

Proposition 19 *We define*

$$q_j = (-1)^{j+1} q$$

and consider $\sigma_j = \mathrm{e}^{\mathrm{i}\,\phi_{j+1}}$ *for* $\phi_j = -2\omega \frac{h_j}{c_j} \in [-\phi, 0]$ *for sufficiently small* $\phi > 0$. *We set*

$$Q_j := \begin{cases} Q_1 & j = 1, \\ \dfrac{(-1)^{j+1}\, q + Q_{j-1}}{\sigma_j \left(1 + (-1)^{j+1}\, q Q_{j-1}\right)} & 2 \leq j \leq n, \end{cases}$$

for some Q_1 *with* $|Q_1| < 1$. *Then the estimate*

$$|Q_j|^2 \leq 1 - (1 - |Q_1|^2)\alpha_q^{\omega/c_{\min}} \tag{47}$$

holds for some $\alpha_q \in (0, 1)$.

Remark 20 $\alpha_q \to 0$ *as* $q \to 1$.

Proof. We recall $Q'_j = \sigma_j Q_j$, $\rho_j := \left|Q'_j\right| = |Q_j|$ and $Q'_j =: s_j \rho_j$ for some $s_j = \mathrm{e}^{\mathrm{i}\,\psi_j}$ with $\psi_j \in [-\pi, \pi[$. We double the iteration and arrive at

$$s_{j+2}\rho_{j+2} = \frac{(1 - \sigma_{j+1})\,(-1)^{j+1}\, q + \left(1 - \sigma_{j+1} q^2\right) \frac{s_j}{\sigma_j} \rho_j}{\sigma_{j+1} - q^2 + (1 - \sigma_{j+1})\,(-1)^j\, q \frac{s_j}{\sigma_j} \rho_j}.$$

Two dimensional Taylor expansion w.r.t (ϕ_j, ϕ_{j+1}) at 0 yields

$$\rho_{j+2}^2 = \rho_j^2 + (-1)^j\, \frac{2q}{1 - q^2}(1 - \rho_j^2)\rho_j\, (\sin\psi_j)\, \phi_{j+1} + \text{h.o.t.} \tag{48}$$

First, we leave away the higher other term ("h.o.t.") in (48), restrict to the case of even j in (48), and study the recursion

$$\tilde{\rho}_{j+1}^2 = \tilde{\rho}_j^2 + \frac{2q}{1 - q^2}\left(1 - \tilde{\rho}_j^2\right) \underbrace{(\sin\psi_{2j})\, \phi_{2j+1} \tilde{\rho}_j}_{=:\gamma_j},$$

for $j \geq 1$. This recursion can be resolved and we obtain

$$\tilde{\rho}_{j+1}^2 = \tilde{\rho}_{j+1}^2 \left((\gamma_k)_{k=1}^j\right) = \tilde{\rho}_1^2 + \frac{2q}{1 - q^2}\left(1 - \tilde{\rho}_1^2\right) \sum_{\ell=1}^{j} \gamma_\ell \prod_{k=1}^{\ell-1} \left(1 - \frac{2q}{1 - q^2}\gamma_k\right), \tag{49}$$

where $\gamma_k \in [-\phi, \phi]$, $\frac{2q}{1-q^2} \in \mathbb{R}_+$ and $\phi > 0$. Now for $\phi \leq \frac{1-q^2}{4q}$ and $i \leq j$, we have

$$\begin{aligned}
\frac{\partial \tilde{\rho}_{j+1}^2}{\partial \gamma_i} =& \frac{2q}{1-q^2}(1-\rho_1^2)\frac{\partial}{\partial \gamma_i}\left(\gamma_i \prod_{k=1}^{i-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right) + \sum_{\ell=i+1}^{j}\gamma_\ell \prod_{k=1}^{\ell-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right)\right) \\
=& \frac{2q}{1-q^2}(1-\rho_1^2)\prod_{k=1}^{i-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right)\left(1-\frac{2q}{1-q^2}\sum_{\ell=i+1}^{j}\gamma_\ell \prod_{k=i+1}^{\ell-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right)\right) \\
=& \frac{2q}{1-q^2}(1-\rho_1^2)\prod_{\substack{k=1\\k\neq i}}^{j}\left(1-\frac{2q}{1-q^2}\gamma_k\right) > 0.
\end{aligned}$$

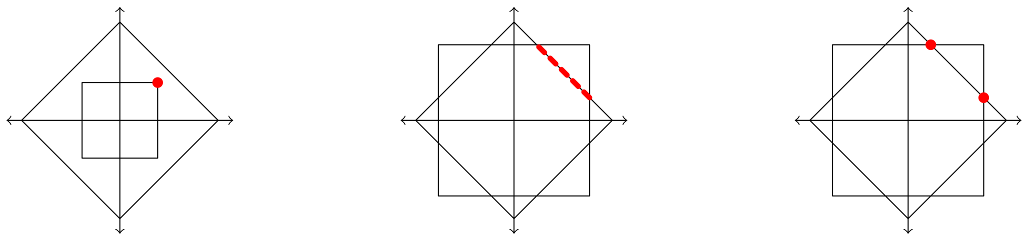
This means that the r.h.s of (49) is increasing in γ_i. We want to find the maximum of (49) under the restriction $\sum_{i=1}^{j}|\gamma_i| < s$, for some $s \in \mathbb{R}_+$. If $s \geq n\phi$, then clearly $\gamma_i = \phi$ for all i is the maximizer (see Figure 2a). Since (49) is increasing in every variable, we can assume $\gamma_i \geq 0$ and consider the restriction $\sum_{i=1}^{j}\gamma_i = s$. Let $s < n\phi$, We consider the $(j-1)$-dimensional plane going through the points $se_1, se_2, \dots se_j$, parametrized by $\tau : \mathbb{R}^{j-1} \to \mathbb{R}^j$, $(x_1, \dots, x_{j-1}) \to s(e_j + (e_1 - e_j)x_1 + (e_2 - e_j)x_2 \cdots + (e_{j-1} - e_j)x_{j-1})$ (see Figure 2b). Then $\tilde{\rho}_j^2 \circ \tau$ is a function defined on $\mathbb{R}^{j-1}$ by

$$\begin{aligned}
\tilde{\rho}_{j+1}^2 \circ \tau\ (\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_{j-1}) =& \tilde{\rho}_{j+1}^2\left(s\tilde{\gamma}_1, \dots, s\tilde{\gamma}_{j-1}, s\left(1-\sum_{i=1}^{j-1}\tilde{\gamma}_i\right)\right) \\
=& \tilde{\rho}_1^2 + \frac{2q}{1-q^2}(1-\tilde{\rho}_1^2)s\sum_{l=1}^{j-1}\tilde{\gamma}_l\prod_{k=1}^{l-1}\left(1-\frac{2q}{1-q^2}s\tilde{\gamma}_k\right) \\
&+ \frac{2q}{1-q^2}(1-\tilde{\rho}_1^2)s\left(1-\sum_{i=1}^{j-1}\tilde{\gamma}_i\right)\prod_{k=1}^{j-1}\left(1-\frac{2q}{1-q^2}s\tilde{\gamma}_k\right).
\end{aligned}$$

The partial derivatives of $\tilde{\rho}_{j+1} \circ \tau$ are given by

$$\frac{\partial}{\partial \tilde{\gamma}_i}\tilde{\rho}_{j+1}^2 \circ \tau = \left(\frac{2q}{1-q^2}\right)^2 s^2(1-\tilde{\rho}_1^2)\prod_{\substack{k=1\\k\neq i}}^{j-1}\left(1-\frac{2q}{1-q^2}s\tilde{\gamma}_k\right)\left(\tilde{\gamma}_i - \left(1-\sum_{k=1}^{j-1}\tilde{\gamma}_k\right)\right).$$

The r.h.s. is zero for all i iff $\tilde{\gamma}_k = \frac{1}{j}$ for all k. Therefore for $(\gamma_k)_{k=1}^{j} = \tau((\tilde{\gamma}_k)_{k=1}^{j-1})$, we have $\gamma_k = \frac{s}{j}$



(a) If $s \geq n\phi$, the maximum is achieved at the vertex $\bullet$.

(b) In the first step, we look for a maximizer along the dashed line.

(c) In the second step, we check if the points at the boundary ($\bullet$) are maximizers.

Figure 2: Showing the different situations for finding the maximum of (49) under the restriction $\sum_{i=1}^{j} |\gamma_i| \leq s$.

for all $k = 1, 2, \ldots, j$, and $\frac{s}{j} \leq \frac{1-q^2}{4q}$, since we also have $\gamma_k \leq \phi \leq \frac{1-q^2}{4q}$. In this case, we compute

$$
\begin{aligned}
\tilde{\rho}_{j+1}^2 &= \rho_1^2 + \frac{2q}{1-q^2}(1-\tilde{\rho}_1^2) \sum_{\ell=1}^{j} \frac{s}{j} \left(1 - \frac{2q}{1-q^2}\frac{s}{j}\right)^{l-1} \\
&= \tilde{\rho}_1^2 + \frac{2q}{1-q^2}(1-\tilde{\rho}_1^2) \frac{s}{j} \left(\frac{1 - \left(1 - \frac{2q}{1-q^2}\frac{s}{j}\right)^{j}}{\frac{2q}{1-q^2}\frac{1}{j}} \right) \\
&= \tilde{\rho}_1^2 + (1-\tilde{\rho}_1^2) \left(1 - \left(1 - \frac{2q}{1-q^2}\frac{s}{j}\right)^{j}\right) \\
&\leq \tilde{\rho}_1^2 + (1-\tilde{\rho}_1^2) \left(1 - \left(\frac{1}{2}\right)^{\frac{4qs}{1-q^2}}\right) \\
&\overset{s \leq \frac{4\omega}{c_{\min}}}{\leq} \tilde{\rho}_1^2 + (1-\tilde{\rho}_1^2) \left(1 - \alpha_q^{\frac{\omega}{c_{\min}}}\right) \\
&= 1 - (1-\tilde{\rho}_1^2)\alpha_q^{\frac{\omega}{c_{\min}}},
\end{aligned}
$$

for some $0 < \alpha_q < 1$. This gives an upper bound on ρ_j as stated in (47). However, the maximizer of $\tilde{\rho}_j \circ \tau$ might also lie on the boundary of the set we are considering. We investigate this in more details. Consider $\tilde{\rho}_j \circ \tau$ restricted to $D \subset \mathbb{R}^{j-1}$ such that $\tau(D)$ is the the intersection of the cube $(0, \phi]^j$ and the set $\{x \in \mathbb{R}^j | \sum_{i=1}^{j} x_i = s\}$ (see Figure 2c). Now we note that on the boundary of this set we have $\gamma_k \in \{0, \phi\}$ for some k. Finding the maximum on the boundary can be reduced to the same problem of dimension $n-2$ (fixing $\gamma_k \in \{0, \phi\}$) with $\tilde{s} = s - \gamma_k$. Since $\tilde{\rho}_j$ is symmetric in all its variables we find that for m s.t. $(m-1)\phi < s \leq m\phi$ a maximizer is given by

$$
\begin{aligned}
\gamma_k &= \phi, && \text{for } k = 1, \ldots, m-1 \\
\gamma_m &= s - (m-1)\phi, \\
\gamma_k &= 0, && \text{for } k > m.
\end{aligned}
$$

With this choice of γ_k, we have

$$
\begin{aligned}
\tilde{\rho}_{j+1}^2 &\le \tilde{\rho}_1^2 + \frac{2q}{1-q^2}\left(1-\tilde{\rho}_1^2\right)\sum_{l=1}^{m}\gamma_l\prod_{k=1}^{l-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right)\\
&\le \tilde{\rho}_1^2 + \frac{2q}{1-q^2}(1-\tilde{\rho}_1^2)\phi\sum_{l=1}^{m}\left(1-\frac{2q}{1-q^2}\phi\right)^{l-1}\\
&= \tilde{\rho}_1^2 + \frac{2q}{1-q^2}\left(1-\tilde{\rho}_1^2\right)\phi\frac{1-(1-\frac{2q}{1-q^2}\phi)^m}{\frac{2q}{1-q^2}\phi} = \tilde{\rho}_1^2 + \left(1-\tilde{\rho}_1^2\right)\left(1-\left(1-\frac{2q}{1-q^2}\phi\right)^m\right) \qquad (50)\\
&\le \tilde{\rho}_1^2 + (1-\tilde{\rho}_1^2)\left(1-\left(1-\frac{2q}{1-q^2}\phi\right)^{s/\phi+1}\right) = 1-(1-\tilde{\rho}_1^2)\left(1-\frac{2q}{1-q^2}\phi\right)^{s/\phi+1}\\
&\le 1-(1-\tilde{\rho}_1^2)\alpha_q^{\omega/c_{\min}},
\end{aligned}
$$

using $\phi < \frac{1-q}{4q}$ and $s \le 4\omega/c_{\min}$, with a possibly adjusted $\alpha_q \in (0,1)$ from (50).
Now we consider the general case including the higher other terms in (48). Let j be even. Taylor expansion w.r.t. (ϕ_j, ϕ_{j+1}) yields

$$
\begin{aligned}
\rho_{j+2}^2 \le \rho_j^2 &+ \frac{2q}{1-q^2}\left(1-\rho_j^2\right)\rho_j\left(\sin\psi_j\right)\phi_{j+1}\\
&+ \frac{2q}{1-q^2}(1-\rho_j^2)K(\phi_j+\phi_{j+1})^2\\
\le \rho_j^2 &+ \frac{2q}{1-q^2}(1-\rho_j^2)(|\phi_{j+1}| + K(|\phi_j|+|\phi_{j+1}|)^2),
\end{aligned}
$$

for a constant $K \in \mathbb{R}_{>0}$ uniformly bounded if $\phi < \frac{1}{8}$. Now we compute

$$
\begin{aligned}
\rho_{j+2}^2 &\le \rho_j^2 + \frac{2q}{1-q^2}(1-\rho_j^2)(|\phi_{j+1}| + K(|\phi_j|+|\phi_{j+1}|)^2)\\
&= \rho_j^2\left(1-\frac{2q}{1-q^2}(\underbrace{|\phi_{j+1}| + K(|\phi_j|+|\phi_{j+1}|)^2}_{=:\eta_j})\right) + \frac{2q}{1-q^2}(|\phi_{j+1}| + K(|\phi_j|+|\phi_{j+1}|)^2)\\
&= \rho_j^2\left(1-\frac{2q}{1-q^2}\eta_j\right) + \frac{2q}{1-q^2}\eta_j.
\end{aligned}
$$

For $\phi < \min\left\{\frac{1}{4K}, \frac{1-q^2}{4q}\right\}$, we have that $(1-\frac{2q}{1-q^2}\eta_j)$ is positive and hence, we can consider the majorant

$$
\rho_{j+2}'^2 = \rho_j'^2\left(1-\frac{2q}{1-q^2}\eta_j\right) + \frac{2q}{1-q^2}\eta_j
$$

for $0 \le \eta_j < \frac{1-q^2}{2q}$ and resolve the representation

$$
\rho_{j+1}'^2 = \rho_1'^2 + \frac{2q}{1-q^2}(1-\rho_1'^2)\sum_{\ell=1}^{j/2}\gamma_\ell\prod_{k=1}^{\ell-1}\left(1-\frac{2q}{1-q^2}\gamma_k\right),
$$

which is increasing in $\gamma_\ell := \eta_{2\ell}$. By the same argument as before we receive

$$\rho_j^2 \leq 1 - (1-\rho_1^2)\alpha_q^{\omega/c_{\min}},$$

for some $0 < \alpha_q < 1$. ■

Proposition 21 *Consider the Helmholtz problem* (13) *with diffusion coefficient $a = 1$, $f = 0$ and boundary values g_1, g_2. Let c be oscillating between $c_{\min}$ and $c_{\max}$. If $h_j \lesssim \frac{1}{\omega}$ is small enough for all j then the stability constant in* (10) *can be bounded independently of the number of jumps by*

$$C_{\text{stab}} \leq C_q \alpha_q^{-\frac{\omega}{c_{\min}}},$$

for some $0 < \alpha_q < 1$ and $C_q \in \mathbb{R}$.

5.4 Stability Estimate

Proposition 18 and 21 can be combined to find a final stability estimate (10) for the Helmholtz problem.

Theorem 22 *Let c be piecewise constant and perfectly oscillating between two values, i.e.*

$$c_j = \begin{cases} c_0 & \text{if } j \text{ is odd,} \\ c_1 & \text{if } j \text{ is even,} \end{cases}$$

with $0 < c_{\min} = \min\{c_0, c_1\} \leq \max\{c_0, c_1\} = c_{\max} < \infty$. Let u be the (weak) solution of the Helmholtz problem

$$-u'' - \left(\frac{\omega}{c}\right)^2 u = 0 \text{ in } \Omega = (-1,1)\,, \tag{51}$$

with boundary conditions

$$\begin{aligned} -u' - \mathrm{i}\,\frac{\omega}{c_1} u &= g_1 \text{ at } x = -1, \\ u' - \mathrm{i}\,\frac{\omega}{c_n} u &= g_2 \text{ at } x = 1. \end{aligned}$$

Then u satisfies

$$\left(\int_\Omega |u'|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2\right)^{\frac{1}{2}} \leq C_{\text{stab}} \max\{|g_1|, |g_2|\},$$

with

$$C_{\text{stab}} \leq C_q \alpha_q^{-\frac{\omega}{c_{\min}}} \quad \text{for some } 0 < \alpha_q < 1 \text{ and } C_q \in \mathbb{R}. \tag{52}$$

The estimate (52) is independent of the number of jumps of c and does not require any periodicity of the media.

5.5 Remarks on the Energy Estimate

We make some remarks on the achieved results.

1. In the previous section, we derived an estimate on the stability constant C_{stab} with constant principal part. However, the case where a is not constant can be reduced to the constant case by introducing the new function
$$u(x) = v \circ \eta(x)$$
with $\eta : (-1,1) \to (-1,1)$
$$\eta(x) = -1 + \frac{2}{A} \int_{-1}^{x} \frac{1}{a(s)} ds \quad \text{and} \quad A = \int_{-1}^{1} \frac{ds}{a(s)}.$$
Then, it is an easy exercise to verify that v satisfies a Helmholtz equation with the Laplacian as its principle part, again with Robin boundary conditions.

2. The result in Theorem 52 is valid for coefficients c which oscillate perfectly between two values. With the developed technique described above the case where c is monotone can be handled similarly to Section 5.2 (but independent of n). For arbitrary piecewise configuration of c, the handling of the sequence (Q_j) requires more technicalities and is an open question.

3. If the configuration of c is *fixed*, one can estimate the growth of $|Q_j|$ with (46) where n is the number of jumps of c, in particular independent of ω (and ε), to show that the stability constant is bounded from above independently of ω. This was also shown in the PhD thesis [8].

6 Proof of the Representation Formulas

Proof of Lemma 5. We consider the ansatz (16). The transmission condition leads to

$$\begin{bmatrix} \mathbf{C}^{(1)} & -\mathbf{G}^{(1)} & \mathbf{0} & \cdots & & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{(2)} & -\mathbf{G}^{(2)} & & & \vdots \\ \vdots & & \ddots & \ddots & & \\ & & & & & \mathbf{0} \\ \mathbf{0} & & & & \mathbf{C}^{(n)} & -\mathbf{G}^{(n)} \end{bmatrix} \begin{pmatrix} A_1 \\ B_1 \\ \vdots \\ A_{n+1} \\ B_{n+1} \end{pmatrix} = \mathbf{0}, \tag{53}$$

where $\mathbf{C}^{(i)}, \mathbf{G}^{(i)} \in \mathbb{R}^{2\times 2}$ are given by

$$\mathbf{C}^{(i)} := \begin{bmatrix} c_{11}^{(i)} & c_{12}^{(i)} \\ c_{21}^{(i)} & c_{22}^{(i)} \end{bmatrix} := \begin{bmatrix} \alpha_{i,i} & \dfrac{1}{\alpha_{i,i}} \\ \dfrac{\alpha_{i,i}}{c_i} & -\dfrac{1}{\alpha_{i,i} c_i} \end{bmatrix},$$

$$\mathbf{G}^{(i)} := \begin{bmatrix} d_{11}^{(i)} & d_{12}^{(i)} \\ d_{21}^{(i)} & d_{22}^{(i)} \end{bmatrix} := \begin{bmatrix} \alpha_{i+1,i} & \dfrac{1}{\alpha_{i+1,i}} \\ \dfrac{\alpha_{i+1,i}}{c_{i+1}} & -\dfrac{1}{\alpha_{i+1,i} c_{i+1}} \end{bmatrix}.$$

Inserting the boundary conditions into (53) results in the following block-tridiagonal system

$$\begin{bmatrix} \mathbf{S}^{(1)} & \mathbf{T}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{R}^{(1)} & \mathbf{S}^{(2)} & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & & & \mathbf{T}^{(n-1)} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{R}^{(n-1)} & \mathbf{S}^{(n)} \end{bmatrix} \begin{pmatrix} B_1 \\ A_2 \\ B_2 \\ \vdots \\ A_n \\ B_n \\ A_{n+1} \end{pmatrix} = \begin{bmatrix} -c_{11}^{(1)} & 0 \\ -c_{21}^{(1)} & \vdots \\ 0 & 0 \\ \vdots & d_{12}^{(n)} \\ 0 & d_{22}^{(n)} \end{bmatrix} \begin{pmatrix} A_1 \\ B_{n+1} \end{pmatrix} \tag{54}$$

with

$$\mathbf{R}^{(i)} = \begin{bmatrix} 0 & c_{11}^{(i+1)} \\ 0 & c_{21}^{(i+1)} \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{i+1,i+1} \\ 0 & \frac{\alpha_{i+1,i+1}}{c_{i+1}} \end{bmatrix},$$
$$\mathbf{S}^{(i)} = \begin{bmatrix} c_{12}^{(i)} & -d_{11}^{(i)} \\ c_{22}^{(i)} & -d_{21}^{(i)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_{i,i}} & -\alpha_{i+1,i} \\ -\frac{1}{\alpha_{i,i}c_i} & -\frac{\alpha_{i+1,i}}{c_{i+1}} \end{bmatrix},$$
$$\mathbf{T}^{(i)} = \begin{bmatrix} -d_{12}^{(i)} & 0 \\ -d_{22}^{(i)} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha_{i+1,i}} & 0 \\ \frac{1}{\alpha_{i+1,i}c_{i+1}} & 0 \end{bmatrix}.$$

Next we transform (54) to a tridiagonal system. Let row (i) denote the i-th row of (54). First, we replace for $1 \le i \le n-1$ the row $(2i)$ by $\rho_i \,\mathrm{row}\,(2i-1) - c_i\rho_i \,\mathrm{row}\,(2i)$. This leads to

$$\begin{bmatrix} \tilde{\mathbf{S}}^{(1)} & \tilde{\mathbf{T}}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \tilde{\mathbf{R}}^{(1)} & \tilde{\mathbf{S}}^{(2)} & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & & & \tilde{\mathbf{T}}^{(n-1)} \\ \mathbf{0} & \dots & \mathbf{0} & \tilde{\mathbf{R}}^{(n-1)} & \tilde{\mathbf{S}}^{(n)} \end{bmatrix} \begin{pmatrix} B_1 \\ A_2 \\ B_2 \\ \vdots \\ A_n \\ B_n \\ A_{n+1} \end{pmatrix} = \begin{bmatrix} -\alpha_{1,1} & & 0 \\ 0 & & \vdots \\ \vdots & & 0 \\ & & \frac{1}{\alpha_{n+1,n}} \\ 0 & \frac{\rho_n}{\alpha_{n+1,n}} & \frac{c_n+c_{n+1}}{c_{n+1}} \end{bmatrix} \begin{pmatrix} A_1 \\ B_{n+1} \end{pmatrix} \tag{55}$$

with

$$\tilde{\mathbf{R}}^{(i)} = \begin{bmatrix} 0 & \alpha_{i+1,i+1} \\ 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{S}}^{(i)} = \begin{bmatrix} \frac{1}{\alpha_{i,i}} & -\alpha_{i+1,i} \\ 2\frac{\rho_i}{\alpha_{i,i}} & \rho_i\alpha_{i+1,i}\left(\frac{c_i-c_{i+1}}{c_{i+1}}\right) \end{bmatrix}, \quad \tilde{\mathbf{T}}^{(i)} = \begin{bmatrix} -\frac{1}{\alpha_{i+1,i}} & 0 \\ -\frac{\rho_i}{\alpha_{i+1,i}}\frac{c_i+c_{i+1}}{c_{i+1}} & 0 \end{bmatrix}.$$

In the next step, we replace for $1 \le i \le n$ the row $(2i-1)$ by $\rho_i \frac{c_i+c_{i+1}}{c_{i+1}}\delta_i \,\mathrm{row}\,(2i-1) - \delta_i \,\mathrm{row}\,(2i)$. This leads to

$$\begin{bmatrix} \hat{\mathbf{S}}^{(1)} & \hat{\mathbf{T}}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \hat{\mathbf{R}}^{(1)} & \hat{\mathbf{S}}^{(2)} & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & & & \hat{\mathbf{T}}^{(n-1)} \\ \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{R}}^{(n-1)} & \hat{\mathbf{S}}^{(n)} \end{bmatrix} \begin{pmatrix} B_1 \\ A_2 \\ B_2 \\ \vdots \\ A_n \\ B_n \\ A_{n+1} \end{pmatrix} = \begin{bmatrix} -\rho_1\frac{c_1+c_2}{c_2}\delta_1\alpha_{1,1} & & 0 \\ 0 & & \vdots \\ \vdots & & 0 \\ & & 0 \\ 0 & \frac{\rho_n}{\alpha_{n+1,n}} & \frac{c_n+c_{n+1}}{c_{n+1}} \end{bmatrix} \begin{pmatrix} A_1 \\ B_{n+1} \end{pmatrix} \tag{56}$$

with

$$\hat{\mathbf{R}}^{(i)} = \begin{bmatrix} 0 & \rho_{i+1}\delta_{i+1}\frac{c_{i+1}+c_{i+2}}{c_{i+2}}\alpha_{i+1,i+1} \\ 0 & 0 \end{bmatrix},$$
$$\hat{\mathbf{S}}^{(i)} = \begin{bmatrix} \frac{\rho_i\delta_i}{\alpha_{i,i}}\frac{c_i - c_{i+1}}{c_{i+1}} & -2\frac{c_i}{c_{i+1}}\delta_i\rho_i\alpha_{i+1,i} \\ 2\frac{\rho_i}{\alpha_{i,i}} & \rho_i\alpha_{i+1,i}\frac{c_i - c_{i+1}}{c_{i+1}} \end{bmatrix},$$
$$\hat{\mathbf{T}}^{(i)} = \begin{bmatrix} 0 & 0 \\ -\frac{\rho_i}{\alpha_{i+1,i}}\frac{c_i + c_{i+1}}{c_{i+1}} & 0 \end{bmatrix}.$$

We choose

$$\delta_i = -\frac{1}{\alpha_{i,i}\alpha_{i+1,i}}\frac{c_{i+1}}{c_i} \quad \text{and} \quad \rho_i = \frac{\alpha_{i+1,i}}{c_i + c_{i+1}}$$

to obtain the following symmetric tridiagonal system of linear equations

$$\underbrace{\begin{bmatrix} \check{\mathbf{S}}^{(1)} & \check{\mathbf{T}}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \left(\check{\mathbf{T}}^{(1)}\right)^{\mathsf{T}} & \check{\mathbf{S}}^{(2)} & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & & & \check{\mathbf{T}}^{(n-1)} \\ \mathbf{0} & \dots & \mathbf{0} & \left(\check{\mathbf{T}}^{(n-1)}\right)^{\mathsf{T}} & \check{\mathbf{S}}^{(n)} \end{bmatrix}}_{=:\check{\mathbf{M}}^{(2n)}} \underbrace{\begin{pmatrix} B_1 \\ A_2 \\ B_2 \\ \vdots \\ A_n \\ B_n \\ A_{n+1} \end{pmatrix}}_{\mathbf{x}^{(2n)}} = \underbrace{\begin{bmatrix} \frac{1}{c_1} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ & 0 \\ 0 & \frac{1}{c_{n+1}} \end{bmatrix}}_{\mathbf{r}^{(2n)}} \begin{pmatrix} A_1 \\ B_{n+1} \end{pmatrix} \tag{57}$$

with the 2×2 matrices

$$\check{\mathbf{S}}^{(i)} := \begin{bmatrix} \frac{c_{i+1}-c_i}{c_{i+1}+c_i}\frac{1}{c_i\alpha_{i,i}^2} & \frac{2}{c_i+c_{i+1}}\frac{\alpha_{i+1,i}}{\alpha_{i,i}} \\ \frac{2}{c_i+c_{i+1}}\frac{\alpha_{i+1,i}}{\alpha_{i,i}} & -\frac{c_{i+1}-c_i}{c_{i+1}+c_i}\frac{\alpha_{i+1,i}^2}{c_{i+1}} \end{bmatrix}, \quad \check{\mathbf{T}}^{(i)} := \begin{bmatrix} 0 & 0 \\ -\frac{1}{c_{i+1}} & 0 \end{bmatrix}.$$

It is easy to see that $\left(\check{\mathbf{M}}^{(2n)}\right)^{-1}$ can be factorized $\left(\check{\mathbf{M}}^{(2n)}\right)^{-1} = \mathbf{D}^{(2n)}\mathbf{M}_{\text{Green}}^{(2n)}\mathbf{D}^{(2n)}$ with $\mathbf{D}^{(2n)}$ and $\mathbf{M}_{\text{Green}}^{(2n)}$ as in (20) and (21).

■

Proof of Lemma 7. First, we will prove

$$\det \mathbf{M}^{(2n)} = (-1)^n \, \tilde{p}_n \tag{58}$$

for $n \geq 1$ by induction with $\tilde{p}_n$ as in (24). It is easy to check that $\det \mathbf{M}^{(2)} = -1 = -\tilde{p}_1$. From now on we assume that (58) holds for $n' = 2, 3, \dots n-1$. From (63) we derive

$$\det \mathbf{M}^{(2n)} = -q_n \det \mathbf{M}^{(2n-1)} - \left(1 - q_n^2\right) \det \mathbf{M}^{(2n-2)}$$
$$\det \mathbf{M}^{(2n-1)} = q_n \det \mathbf{M}^{(2n-2)} - \frac{1}{\sigma_{n-1}} \det \mathbf{M}^{(2n-3)}.$$

and, in turn, we get

$$\det \mathbf{M}^{(2n)} = -\det \mathbf{M}^{(2n-2)} + \frac{q_n}{\sigma_{n-1}} \sum_{\ell=1}^{n-1} \frac{(-1)^{\ell+1} q_{n-\ell}}{\prod_{k=2}^{\ell} \sigma_{n-k}} \det \mathbf{M}^{(2(n-\ell-1))}, \tag{59a}$$

$$\det \mathbf{M}^{(2n-1)} = \sum_{\ell=1}^{n} (-1)^{\ell+1} \frac{q_{n+1-\ell}}{\prod_{k=n+1-\ell}^{n-1} \sigma_k} \det \mathbf{M}^{(2n-2\ell)}. \tag{59b}$$

We insert the induction assumption (58) for $\tilde{p}_n$ into the right-hand side of (59a). Then (58) follows if we prove

$$\tilde{p}_n = \tilde{p}_{n-1} + \frac{q_n}{\sigma_{n-1}} \sum_{\ell=1}^{n-1} \frac{q_{n-\ell}}{\prod_{k=2}^{\ell} \sigma_{n-k}} \tilde{p}_{n-\ell-1} \tag{60}$$

for $n \geq 1$. It is simple to verify this equality for $n = 1$ and we assume for the following that this holds for $n' = 1, 2, \ldots, n-1$.

For $1 \leq k \leq n-1$, we set

$$\delta_{n,k} := \sum_{\ell=k}^{n-1} \frac{q_{n-\ell}}{\prod_{j=2}^{\ell} \sigma_{n-j}} \tilde{p}_{n-\ell-1}$$

and prove

$$\delta_{n,k} = \frac{\prod_{j=1}^{n-k} \sigma_j}{\prod_{j=1}^{n-2} \sigma_j} \left(\prod_{j=1}^{n-1-k} (1 + q_{j+1} Q_j) \right) Q_{n-k} \tag{61}$$

by induction over $k = n-1, n-2, \ldots 1$. We denote the right-hand side in (61) by $\tilde{\delta}_{n,k}$ and prove $\delta_{n,k} = \tilde{\delta}_{n,k}$. For $k = n-1$ we have

$$\tilde{\delta}_{n,n-1} = \frac{q_1}{\prod_{j=1}^{n-2} \sigma_j} \qquad \delta_{n,n-1} = \frac{q_1}{\prod_{k=2}^{n-1} \sigma_{n-k}}.$$

Assume the assertion holds for $k' = n-1, n-2, \ldots k+1$. Then

$$\begin{aligned}
\delta_{n,k} &= \frac{q_{n-k}}{\prod_{j=2}^{k} \sigma_{n-j}} \tilde{p}_{n-k-1} + \delta_{n,k+1} \\
&\overset{(61)}{=} \frac{q_{n-k}}{\prod_{j=2}^{k} \sigma_{n-j}} \tilde{p}_{n-k-1} + \frac{\prod_{j=1}^{n-k-1} \sigma_j}{\prod_{j=1}^{n-2} \sigma_j} \left(\prod_{j=1}^{n-2-k} (1 + q_{j+1} Q_j) \right) Q_{n-1-k} \\
&\overset{\text{ind.}}{=} \prod_{\ell=1}^{n-k-2} (1 + q_{\ell+1} Q_\ell) \left(\frac{q_{n-k}}{\prod_{j=2}^{k} \sigma_{n-j}} + \frac{\prod_{j=1}^{n-k-1} \sigma_j}{\prod_{j=1}^{n-2} \sigma_j} Q_{n-1-k} \right) \\
&= \frac{\prod_{\ell=1}^{n-k-2} (1 + q_{\ell+1} Q_\ell)}{\prod_{j=2}^{k} \sigma_{n-j}} (q_{n-k} + Q_{n-1-k}) \\
&= \frac{\prod_{\ell=1}^{n-k-2} (1 + q_{\ell+1} Q_\ell)}{\prod_{j=2}^{k} \sigma_{n-j}} Q_{n-k} (1 + q_{n-k} Q_{n-1-k}) \sigma_{n-k} = \tilde{\delta}_{n,k}.
\end{aligned}$$

Hence, (61) is proved. We insert this into the right-hand side of (60) and get

$$\begin{aligned}\left(\prod\nolimits_{\ell=1}^{n-2}(1+q_{\ell+1}Q_\ell)\right)+\frac{q_n}{\sigma_{n-1}}\delta_{n,1}&=\left(\prod\nolimits_{\ell=1}^{n-2}(1+q_{\ell+1}Q_\ell)\right)\left(1+\sigma_{n-1}\frac{q_n}{\sigma_{n-1}}Q_{n-1}\right)\\&=\left(\prod\nolimits_{\ell=1}^{n-2}(1+q_{\ell+1}Q_\ell)\right)(1+q_nQ_{n-1})\\&=\prod\nolimits_{\ell=1}^{n-1}(1+q_{\ell+1}Q_\ell)=p_n.\end{aligned}$$

It remains to prove (59b). We insert (58) into the right-hand side of (59b) and employ definition (24)

$$\det\mathbf{M}^{(2n-1)}=(-1)^{n+1}\sum_{\ell=1}^{n}\frac{q_{n+1-\ell}}{\prod_{k=n+1-\ell}^{n-1}\sigma_k}\prod\nolimits_{k=1}^{n-\ell-1}(1+q_{k+1}Q_k).$$

We will prove

$$\det\mathbf{M}^{(2n-1)}=-\sigma_nQ_n\det\mathbf{M}^{(2n)}.$$

For $1\le k\le n$, we introduce the partial sums

$$\lambda_{n,k}:=\sum_{\ell=k}^{n}\frac{q_{n+1-\ell}}{\prod_{j=n+1-\ell}^{n-1}\sigma_j}\prod\nolimits_{j=1}^{n-\ell-1}(1+q_{j+1}Q_j)$$

so that $\det\mathbf{M}^{(2n-1)}=(-1)^{n+1}\lambda_{n,1}$. By induction for $k=n,n-1,\dots,1$ we prove

$$\lambda_{n,k}=\sigma_{n+1-k}\frac{\prod_{j=1}^{n-k}(1+q_{j+1}Q_j)}{\prod_{j=n+1-k}^{n-1}\sigma_j}Q_{n-k+1}. \tag{62}$$

We denote the right-hand side in (62) by $\tilde\lambda_{n,k}$ and prove $\lambda_{n,k}=\tilde\lambda_{n,k}$. For $k=n$, it holds

$$\tilde\lambda_{n,n}=\frac{Q_1}{\prod_{\ell=2}^{n-1}\sigma_\ell}=\frac{q_1}{\prod_{\ell=1}^{n-1}\sigma_\ell}=\lambda_{n,n}.$$

We assume that (62) holds for $k'=n,n-1,\dots,k+1$. Hence,

$$\begin{aligned}\lambda_{n,k}&=\frac{q_{n+1-k}}{\prod_{j=n+1-k}^{n-1}\sigma_j}\prod\nolimits_{j=1}^{n-k-1}(1+q_{j+1}Q_j)+\lambda_{n,k+1}\\&\overset{\text{(ind.)}}{=}\frac{q_{n+1-k}}{\prod_{j=n+1-k}^{n-1}\sigma_j}\prod\nolimits_{j=1}^{n-k-1}(1+q_{j+1}Q_j)+\frac{\prod_{\ell=1}^{n-k-1}(1+q_{\ell+1}Q_\ell)}{\prod_{\ell=n-k+1}^{n-1}\sigma_\ell}Q_{n-k}\\&=\frac{\prod_{j=1}^{n-k-1}(1+q_{j+1}Q_j)}{\prod_{j=n+1-k}^{n-1}\sigma_j}(q_{n+1-k}+Q_{n-k})\\&\overset{(23)}{=}\frac{\prod_{j=1}^{n-k-1}(1+q_{j+1}Q_{j-1})}{\prod_{j=n+1-k}^{n-1}\sigma_j}(1+q_{n-k+1}Q_{n-k})\,Q_{n-k+1}\sigma_{n+1-k}\\&=\tilde\lambda_{n,k}\end{aligned}$$

and this proves (62). We insert this into (62) to obtain

$$\det\mathbf{M}^{(2n-1)}=(-1)^{n+1}\lambda_{n,1}=-\sigma_nQ_n\det\mathbf{M}^{(2n)}.$$

■

A Some Basic Facts from Linear Algebra

Note that the determinant of any $n \times n$ symmetric tridiagonal matrix

$$\mathbf{W}_n := \begin{pmatrix} \gamma_1 & \beta_1 & 0 & 0 \\ \beta_1 & \gamma_2 & \ddots & 0 \\ 0 & \ddots & \ddots & \beta_{n-1} \\ 0 & 0 & \beta_{n-1} & \gamma_n \end{pmatrix}$$

satisfies the three-term recursion

$$\det \mathbf{W}_n = \gamma_n \det \mathbf{W}_{n-1} - \beta_{n-1}^2 \det \mathbf{W}_{n-2}. \tag{63}$$

Let $\mathbf{W}_n^{(i,j)}$ denote the matrix which arises when removing the i-th row and the j-th column. Then,

$$\det \mathbf{W}_n^{(i,n)} = \left(\prod_{\ell=i}^{n-1} \beta_\ell \right) \det \mathbf{W}_{i-1}. \tag{64}$$

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